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SINGULARITY AND NEAR SINGULARITY IN NUMERICAL ANALYSIS*

GEORGE E. FORSYTHE, Stanford University

1. Introduction. In a number of areas of pure mathematics there is one formula or technique for the *regular* case, and a different formula or technique for the *singular* case. The author feels that from the common-sense standpoint of computing, such a sharp dichotomy between the regular and the singular is unlikely to make sense: there must be an important transition zone of the *nearly singular*. It is the purpose of this note to illustrate this idea with several examples. It turns out, moreover, that the nearly-singular case often cannot be solved accurately with the regular method. A sufficiently accurate treatment of nearly-singular problems may usually be presented in one of two ways: (i) by simply applying the method for a close-by singular problem, accepting the error caused by the change of problem; (ii) by recasting the method for the regular case in a form which converges to the method for the singular case, as the problem becomes singular. Examples of both (i) and (ii) will be given. Clearly (ii) is more desirable since it permits the nearly-singular problem to be solved directly and more accurately.

In addition to the rather special matrix problem of Section 7, one could profitably discuss singularity and near singularity of matrices themselves. Such a discussion would be too long and too technical for the present paper.

2. An elementary integral. For $\epsilon \neq 0$ one has the integration formula

$$\int_1^x \frac{dt}{t^{1+\epsilon}} = \frac{1 - x^{-\epsilon}}{\epsilon} \qquad \text{(REGULAR)}. \tag{1}$$

For the corresponding singular case $\epsilon = 0$,

$$\int_1^x \frac{dt}{t} = \log x \qquad \text{(SINGULAR)}. \tag{2}$$

The formulas (1, 2) represent a typical dichotomy between regular and singular cases. To understand the nearly-singular case (where ϵ is near 0), let us evaluate (1) for $x = 2$ and $\epsilon = 0.0001$. We first need $2^{-.0001}$. Using 6-place logarithms, we find

$$\log_{10} 2 = 0.301030,$$

$$\log_{10} 2^{-.0001} = -0.00003\ 01030 = -1 + .99996\ 98970.$$

* The author's work on this paper has been sponsored by various agencies of the United States: at The Ramo-Wooldridge Corp., by the Air Force; at the University of California, Los Angeles, by the Office of Naval Research; at Stanford University, by the National Science Foundation. Reproduction in whole or in part is permitted for any purpose of the United States Government.

Hence $2^{-.0001}=0.99993$, and the integral in (1) becomes $(1-0.99993)(.0001)^{-1}=0.7$, a value correct to only one significant digit.

Every computing person would use some device to evaluate $x^{-\epsilon}$ for ϵ near 0. For example,

$$x^{-\epsilon} = e^{-\epsilon \log x} = 1 - \epsilon \log x + \tfrac{1}{2}(\epsilon \log x)^2 + \cdots,$$

whence

$$\epsilon^{-1}[1 - x^{-\epsilon}] = \log x - \tfrac{1}{2}\epsilon(\log x)^2 + \tfrac{1}{6}\epsilon^2(\log x)^3 - \cdots. \tag{3}$$

From (3) one readily finds for $\epsilon=.0001$, $x=2$, that the integral in (1) equals 0.693137.

We may consider (3) to consist of an expansion of (1) into a series consisting of the answer $\log x$ to the singular problem, plus correction terms. That is, the nearly-singular problem (1) is effectively solved by a formula which is continuous at $\epsilon=0$. On the other hand, formula (3) is useful only for comparatively small values of $|\epsilon \log x|$. Thus (3) illustrates method (ii) of Section 1 in part, but there remains the question of making a smooth transition from representation (3) to representation (1) of the integral, as $|\epsilon \log x|$ grows.

It should be noted that, lacking the series (3), one could solve (1) roughly in the nearly-singular case $\epsilon=.0001$ by simply setting $\epsilon=0$ and adopting the singular answer $\log 2=0.693149$. While correct only to 4 decimals, this value is far superior to 0.7. This paragraph illustrates method (i) of Section 1.

3. Another integral. Dr. Morris Weisfeld of The Ramo-Wooldridge Corp. dealt with an integration problem which illustrates the same principle. To simplify the problem, consider that we are to evaluate

$$\phi(\alpha) = \int_0^1 s\epsilon^{-\alpha s}ds \qquad (0 \leqq \alpha \leqq 1) \tag{4}$$

for the purpose of later forming $\int_0^1 \phi(\alpha)d\alpha$. We see from elementary calculus that for $\alpha \neq 0$

$$\phi(\alpha) = -\frac{\epsilon^{-\alpha}}{\alpha} - \frac{\epsilon^{-\alpha}}{\alpha^2} + \frac{1}{\alpha^2} \qquad \text{(REGULAR)}. \tag{5}$$

while for $\alpha=0$

$$\phi(0) = \tfrac{1}{2} \qquad \text{(SINGULAR)}.$$

The difficulty is that of evaluating $\phi(\alpha)$ from (5) for α near 0, the nearly-singular case. On casual inspection, (5) looks as though $\lim_{\alpha\to 0}\phi(\alpha)$ might not exist, but a brief study of (4) assures us of the continuity of $\phi(\alpha)$ at 0. But certainly (5) is not well suited to computing $\phi(\alpha)$ for α near 0.

The cure is to rewrite (5) as

$$\phi(\alpha) = e^{-\alpha}\left\{\frac{e^{\alpha} - 1 - \alpha}{\alpha^2}\right\}, \tag{6}$$

a form which shows the continuity of $\phi(\alpha)$ at 0. The use of (6) for computation involves the use of readily available tables of $e^{-\alpha}$, and also of some expansion of $(e^{\alpha}-1-\alpha)\alpha^{-2}$ near $\alpha=0$, in analogy with (3).

The point here is that the form (6), suggested by the knowledge that $\phi(0)=\frac{1}{2}$, is a far more useful formula for α near 0 than the formula (5), which is excellent for $\alpha \gg 0$.

4. A differential equation. As a slightly more advanced problem consider the linear differential equation with constant coefficients

$$y''(x) + c_1 y'(x) + c_2 y(x) = 0, \tag{7}$$

to be solved with initial conditions $y(0) = y_0$, $y'(0) = y_0'$.

The text book solution of (7) starts by finding the roots λ_1, λ_2 of the characteristic equation $r^2+c_1r+c_2=0$. If $\lambda_1 \neq \lambda_2$, one has the basis Y_1, Y_2 of solutions of (7), where

$$Y_1(x) = e^{\lambda_1 x}, \qquad Y_2(x) = e^{\lambda_2 x} \qquad \text{(REGULAR)}.$$

However, if $\lambda_1=\lambda_2=\lambda$ one gets a totally different basis Z_1, Z_2:

$$Z_1(x) = e^{\lambda x}, \qquad Z_2(x) = xe^{\lambda x} \qquad \text{(SINGULAR)}.$$

The further solution of the problem in the regular case is to form the general solution $y(x)=A_1Y_1(x)+A_2Y_2(x)$. Bringing in the initial conditions, we must solve for A_1, A_2 the system

$$A_1 + A_2 = y_0, \qquad \lambda_1 A_1 + \lambda_2 A_2 = y_0', \tag{8}$$

whose determinant is $\lambda_2-\lambda_1$.

If $\lambda_1 \approx \lambda_2$, the solution of (8) is bound to be inaccurate. In such a nearly-singular case we are trying to resolve a function $y(x)$ in terms of two nearly-equal functions $Y_1(x)$, $Y_2(x)$.

On the other hand, when $\lambda_1=\lambda_2=\lambda$ we write $y(x)=B_1Z_1(x)+B_2Z_2(x)$, and it is easy to solve the problem. The equations analogous to (8) are

$$B_1 = y_0, \qquad \lambda B_1 + B_2 = y_0', \tag{9}$$

with determinant 1.

Consideration of the singular case suggests to us that when $\lambda_1 \approx \lambda_2$ the basis Y_1, Y_2 must be replaced by a basis which has the property, like Z_1, Z_2, of not consisting of nearly-equal functions. It is proposed that one use the basis

$$U_1(x) = \frac{1}{2}(e^{\lambda_1 x} + e^{\lambda_2 x}), \qquad U_2(x) = \frac{e^{\lambda_1 x} - e^{\lambda_2 x}}{\lambda_1 - \lambda_2}.$$

Introducing the notation $\frac{1}{2}(\lambda_1+\lambda_2)=\mu$, and $\frac{1}{2}(\lambda_1-\lambda_2)=\nu$, we can write

$$U_1(x) = e^{\mu x}\cosh \nu x, \qquad U_2(x) = e^{\mu x}\frac{\sinh \nu x}{\nu}.$$

In the latter form it is clear that, as $\lambda_1\to\lambda$, $\lambda_2\to\lambda$ (and $\mu\to\lambda$ and $\nu\to 0$), we have $U_1(x)\to Z_1(x)$ and $U_2(x)\to Z_2(x)$. The basis U_1, U_2 is thus very well-behaved for the nearly-singular problems where $\lambda_1\approx\lambda_2$.

It may be noted that to deal with $U_1(x)$, $U_2(x)$ we need tables of $e^{\mu x}$, $\cosh \nu x$, and $\nu^{-1}\sinh \nu x$. Of these, only the last function is unusual, and it can be evaluated readily by a Taylor series.

5. Newton's root finder. Let $f(z)$ be an analytic function of the complex variable z, whose zeros are sought. The Newton root-solving process, taught in elementary calculus for real z, is immediately applicable to complex z. One picks some z_1, and for $i=1, 2, \cdots$ one computes

$$z_{i+1} = z_i - f(z_i)/f'(z_i). \tag{10}$$

If $f(a)=0$, if $f(z)$ is analytic at a, and if $|z_1-a|$ is small enough, then it can be shown that $z_i\to a$, as $i\to\infty$. (See Householder [4].)

We repeat an old definition stated, for example, in [5], where Schröder gave a most interesting treatment of Newton's and related algorithms.

DEFINITION. *Given any sequence* $\{z_i\}$ *with limit* a. *If*

$$\lim_{i\to\infty}\frac{z_{i+1}-a}{(z_i-a)^p}=c,$$

where $0<|c|<\infty$, *the convergence of* z_i *to* a *is said to be of order* p.

One usually speaks of *linear* convergence when $p=1$ (for any linear convergence we need $|c|\leqq 1$), *quadratic* convergence when $p=2$, *etc.*

If a is a simple zero of $f(z)$ (*i.e.*, if $f(a)=0$ but $f'(a)\neq 0$), then the convergence of (10), if it takes place, is at least quadratic. In fact, if $z_i\to a$, it will be shown below that

$$\frac{z_{i+1}-a}{(z_i-a)^2}\to\frac{f''(a)}{2f'(a)}=c, \qquad (\text{as } z_i\to a). \tag{11}$$

Thus if $f''(a)\neq 0$, we have precisely quadratic convergence. Quadratic convergence is desirable because it assures us that ultimately the number of correct decimal digits in z_i will approximately double at each step.

The proof of (11) depends on Taylor's expansion of $f(z)$ at z_i in powers of $z-z_i$:

$$f(a) = f(z_i) + f'(z_i)(a-z_i) + \tfrac{1}{2}f''(z_i)(a-z_i)^2 + o(a-z_i)^2 \qquad (\text{as } z_i\to a). \tag{12}$$

From (10), since $f(a)=0$, one finds that

$$\frac{z_{i+1}-a}{(z_i-a)^2}=\frac{f'(z_i)(z_i-a)-f(z_i)+f(a)}{(z_i-a)^2f'(z_i)}. \tag{13}$$

By substituting for $f(a)$ in (13) from (12), and letting $z_i \to a$, one proves (11).

However, if a is a double zero, $f'(a)=0$ but $f''(a)\neq 0$. It then follows from (12) that

$$\frac{z_{i+1}-a}{z_i-a}\to\frac{1}{2} \qquad (\text{as } z_i\to a), \tag{14}$$

representing only linear convergence, so that ultimately one gains only one binary digit per step.

From (14) Schröder proposed [5] that near a double zero a one should use a *doubly-relaxed* modification of Newton's process, whereby*

$$z_{i+1}=z_i-2f(z_i)/f'(z_i). \tag{15}$$

From (15) it can be shown that for a double zero

$$\frac{z_{i+1}-a}{(z_i-a)^2}\to\frac{f'''(a)}{6f''(a)} \qquad (\text{as } z_i\to a). \tag{16}$$

Thus quadratic convergence has been restored to Newton's process for a double zero by the replacement of (10) by (15).

For an n-fold zero a, the corresponding formulas are

$$z_{i+1}=z_i-nf(z_i)/f'(z_i) \tag{17}$$

and

$$\frac{z_{i+1}-a}{(z_i-a)^2}\to\frac{f^{(n+1)}(a)}{n(n+1)f^{(n)}(a)} \qquad (\text{as } z_i\to a), \tag{18}$$

but we shall here confine our attention to simple and double zeros. Formula (18) seems to be due to Ostrowski (unpublished).

In recapitulation, we have two forms of Newton's process:

$$z_{i+1}=z_i-f(z_i)/f'(z_i) \qquad \text{(REGULAR)}, \tag{10}$$

and

$$z_{i+1}=z_i-2f(z_i)/f'(z_i) \qquad \text{(SINGULAR)}. \tag{15}$$

Once again we have a sharp distinction between the regular and the singular cases, and again intuition tells us that there should be cases in between. Is it then possible that there are zeros of analytic functions *intermediate between simple and double*?

* The author wishes to thank his former colleague Professor Henrici for mentioning the importance of double relaxation to him.

To study the situation, we shall consider that $f(z)$ is a quadratic polynomial in z with distinct zeros. Without loss of generality let the zeros be -1 and 1 and take $f(z)=z^2-1$. To simplify notation we shall denote z_i by z and z_{i+1} by z'. Then the classical Newton process (10) can be written in the following two equivalent forms:

$$z' = (1/2)(z + (1/z)), \tag{19}$$

$$z' - 1 = (1/2z)(z - 1)^2. \tag{20}$$

From (20) it is clear that for z near 1, we get the quadratic behavior of (11). But for $|z|$ very large, one sees from (19) that $z' \approx \frac{1}{2}z$, and we have at first the linear behavior represented in the limit by (14). We now understand that, speaking anthropomorphically, the pair of zeros looks like a double zero to Newton's process, when seen from a long distance away. The Newton formula is not powerful enough to be able to resolve two zeros until z gets into their general vicinity. This *resolving power* is the key to the transition between simple and double zeros.

One phenomenon appears in (19) which was impossible for a true double zero—that if $z \approx 0$, then $z' \approx \infty$; *i.e.*, the flat region where $z \approx 0$ has a strong repulsion in Newton's method.

The doubly-relaxed Newton process here yields

$$z' = 1/z. \tag{21}$$

Clearly (21) is able to bring us in one step from $z \approx \infty$ to $z \approx 0$, something quite impossible for (19). (That we are then likely to run into the repulsive region $z \approx 0$ is beside the point.)

We have demonstrated that the classical Newton process will have the behavior of linear convergence, whether the zeros are actually simple or multiple, until z is close enough to a zero for the algorithm to be able to "see" it as a separate entity. We also noted that one cure for the slow convergence is to use an overrelaxation suggested by (15) or (17).

To make a more detailed study it would be well to introduce an *overrelaxation factor* ω suggested by Young [7] in another setting; *i.e.*, let the *overrelaxed Newton process* be defined by

$$z_{i+1} = z_i - \omega f(z_i)/f'(z_i), \tag{22}$$

where in general we select $\omega > 0$ as best we can.

For the function $f(z)=z^2-1$, the algorithm (22) becomes

$$z' = (1 - (\omega/2))z + (\omega/2z).$$

Hence

$$z' - 1 = (1 - (\omega/2) - (\omega/2z))(z - 1). \tag{23}$$

Assume z is in the right half plane $\operatorname{Re} z \geqq 0$, and that we are aiming for the

root $z=1$. For each z we can define the (real) *optimum overrelaxation factor* $\omega_{opt}=W(z)$ by the requirement, taken from (23), that

$$|1-(\omega/2)-(\omega/2z)| = \text{minimum.}$$

This is equivalent to requiring that

$$|\omega(1+(1/z))-2| = \text{minimum.} \tag{24}$$

For $z=x$ (real), it is seen that $W(x)=2x/(1+x)$ (x real). Thus, for $1\leqq x<\infty$, one has $1\leqq W(x)<2$. We therefore get a continuous transition from double relaxation ($\omega=2$) to the classical Newton process ($\omega=1$). We might even say that the zero-pair $\{-1, 1\}$ forms a *zero of order* $W(x)$, as seen from the point $z=x$.

For complex z the computation of $W(z)$ is more difficult. If we admitted complex values of ω, then one would have precisely $W(z)=2z/(1+z)$. But in practice it would probably be sufficiently difficult to adjust ω over real values. Moreover, for application to a general function f probably the choice of ω should be a function only of the quantities $f(z)$, $f'(z)$, and z.

It seems to the author that some profitable research in numerical analysis might be directed towards taking the proper advantage of overrelaxation in Newton's process and in related iterations. The importance would be greatest when the computation of $f(z)$ or $f'(z)$ is relatively costly. This occurs, for example, when $f(z)=|A_0+A_1z+\cdots+A_mz^m|$, where the A_μ are large matrices.

6. Aitken's δ^2 process. In iterative computation of matrix eigenvalues by the "power method," Aitken [1] proposes the use of an acceleration scheme now usually known as the "δ^2 process." Let A be a matrix of order n, with eigenvalues λ_i so numbered that $|\lambda_1|\geqq|\lambda_2|\geqq=\cdots\geqq|\lambda_n|$. One selects an arbitrary vector v_0, and forms $v_1=c_1Av_0, v_2=c_2Av_1, \cdots$. Here one picks the normalizing scalars c_1, $c_2, \cdots$, so that, for example, the first component of each v_k is 1. Let y_k denote some component of v_k, say the rth.

If

$$|\lambda_1| > |\lambda_2|, \tag{25}$$

it is known that the sequence $\{y_k\}$ is convergent and that $\lim_{k\to\infty} y_k=y$ is the rth component of the unique eigenvector of A belonging to λ_1. The difficulty is that the convergence of y_k to y is very slow in many cases, so that too much machine time is consumed in estimating y. As a cure Aitken advocated use of the δ^2 acceleration process.

Suppose $v_0=\beta_1x_1+\beta_2x_2+\cdots$, and assume $\beta_1\neq0$, $\beta_2\neq0$, $|\lambda_2|>|\lambda_3|$. Then

$$\begin{aligned} A^kv_0 &= \beta_1\lambda_1^kx_1+\beta_2\lambda_2^kx_2+o(\lambda_2^k) \\ &= \beta_1\lambda_1^k[x_1+(\beta_2/\beta_1)(\lambda_2/\lambda_1)^kx_2]+o(\lambda_2^k) \qquad (\text{as } k\to\infty). \end{aligned} \tag{26}$$

It follows readily (see Wilkinson [6], for example) that

$$y_k = y + C\mu^k + o(\mu^k) \qquad (\text{as } k \to \infty), \tag{27}$$

where $\mu = \lambda_2/\lambda_1$, and where C is some constant depending on A and v_0. It is therefore usual to regard the sequence $\{y_k\}$ as having the asymptotic behavior of the sequence $\{\eta_k\}: \eta_k = \eta + C\mu^k$, where C is unknown, and to use an acceleration process which would yield the number η exactly from three iterates η_{k-1}, η_k, η_{k+1}. From the last formula it follows that η is equal to ζ_k, where

$$\zeta_k = \frac{\eta_{k-1}\eta_{k+1} - \eta_k^2}{\eta_{k-1} - 2\eta_k + \eta_{k+1}}.$$

Hence Aitken proposes that one take the value

$$z_k = \frac{y_{k-1}y_{k+1} - y_k^2}{y_{k-1} - 2y_k + y_{k+1}} \qquad (\text{REGULAR}) \tag{28}$$

as an approximation to the limit y of the sequence $\{y_k\}$.

It can be shown for the matrix application that asymptotically, as $k \to \infty$, $z_k - y = o(y_k - y)$, so that the δ^2 process is really useful for sufficiently large k, under the hypothesis (25). Nevertheless, for moderate values of k, even when (25) holds, the acceleration is not always very successful in matrix work.

In order to study a possible difficulty, let us examine the sequence y^k when $\lambda_1 = \lambda_2$ is a double eigenvalue with a non-diagonal Jordan canonical block

$$\begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}.$$

This may be regarded as a limiting singular case of a close pair of distinct eigenvalues λ_1, λ_2. Define the *principal vectors* x_1, x_2 by the relations

$$Ax_1 = \lambda_1 x_1, \qquad Ax_2 = \lambda_1 x_2 + x_1. \tag{29}$$

Suppose $v_0 = \beta_1 x_1 + \beta_2 x_2 + \cdots$, and assume $\beta_1 \neq 0$, $\beta_2 \neq 0$, $|\lambda_1| > |\lambda_3|$. It is easy to show from the Jordan canonical form and (29) that

$$\begin{aligned} A^k v_0 &= \beta_1 \lambda_1^k x_1 + k\beta_2 \lambda_1^{k-1} x_1 + \beta_2 \lambda_1^k x_2 + o(\lambda_1^k) \\ &= \lambda_1^{k-1}\left[(\beta_1\lambda_1 + k\beta_2)x_1 + \beta_2\lambda_1 x_2\right] + o(\lambda_1^k) \\ &\simeq (\beta_1\lambda_1 + \beta_2 k)\lambda_1^{k-1}\left[x_1 + \frac{\beta_2/\beta_1}{1 + (k\beta_2/\lambda_1\beta_1)}\, x_2\right] + o(\lambda_1^k) \quad (\text{as } k \to \infty). \end{aligned} \tag{30}$$

Thus $A^k v_0$ converges in direction to x_1. But the asymptotic behavior of a component y_k of v_k is no longer that of (27). Instead, from (30) one can see that

$$y_k = y + \frac{c_1}{c_2 + k} + o\left(\frac{c_1}{c_2 + k}\right) \qquad (\text{as } k \to \infty), \tag{31}$$

where c_1, c_2 depend on A but not on v_0.

As before, let us therefore assume that $\{y_k\}$ has the asymptotic behavior of $\{\eta_k\}$, where

$$\eta_k = \eta + \frac{c_1}{c_2 + k}. \tag{32}$$

From (32) a straightforward calculation shows that

$$\zeta_k = \frac{\eta_{k-1}\eta_{k+1} - \eta_k^2}{\eta_{k-1} - 2\eta_k + \eta_{k+1}} = \eta + \frac{c_1}{2(c_2 + k)}. \tag{33}$$

Thus $\eta = \eta_k + 2(\zeta_k - \eta_k)$, so that to *obtain* η *from* $\{\eta_k\}$ *involves a double relaxation of the* δ^2 *process.*

Hence, returning to the $\{y_k\}$, it seems desirable to estimate y by the formula

$$w_k = y_k + 2(z_k - y_k) \qquad \text{(SINGULAR)}, \tag{34}$$

where z_k is given by (28).

Once more we have a discontinuity between the correct formula for the regular case of a single dominant eigenvalue, and for the singular case of a double dominant eigenvalue with a nondiagonal canonical block. (A double eigenvalue with a diagonal canonical block will behave like a single eigenvalue in the power method.)

As in Section 5, there are probably cases of distinct eigenvalues λ_1, λ_2 for which the singular formula (34) would be preferable to the regular formula (28). It would take a more detailed analysis of the machine representations of (26) and (30) to delimit such almost-singular cases precisely. But, whereas truly double eigenvalues are undoubtedly rare, relative closeness of a pair λ_1, λ_2 occurs often.

When the power method for computing eigenvalues converges slowly, it is normally a signal that $|\lambda_1| \approx |\lambda_2|$. If λ_1 and λ_2 are nevertheless very different, an appropriate small change of A to $A - pI$ will make $|\lambda_1 - p|$ and $|\lambda_2 - p|$ very different, and should improve the rate of convergence enough to permit use of the regular acceleration formula (28). However, if λ_1 is close to λ_2 the shift of origin will not improve the slow convergence. Such a continued slow convergence should suggest the possibility of using the singular acceleration formula (34).

In analogy with the discussion of Section 5, one might introduce an *overrelaxed* δ^2 *process* $w_k = y_k + \omega(z_k - y_k)$, where z_k is given by (28), and where $1 \leqq \omega \leqq 2$. Some numerical experiments might determine how to vary ω so as to speed the completion of the power method.

7. Eigenvectors of a tridiagonal matrix. Other examples of the singular and the nearly singular occur in matrix problems. We shall consider one example drawn from the work of Givens [3].

To compute the eigenvalues and eigenvectors of a real symmetric matrix A, Givens first reduces the problem to that for a tridiagonal matrix A, meaning that $a_{ij}=0$ for $|i-j|>1$. He then finds the eigenvalues λ_k of A by a process which need not be reproduced here. In the next stage, on which we focus our attention, we have an eigenvalue λ of A, and want to find a corresponding eigenvector x. The process considered is to solve the linear system

$$Bx = 0, \tag{35}$$

where B is the tridiagonal matrix $A-\lambda I$ with zero determinant. (Recall that A and B have the same eigenvector system.) Writing (35) out in detail, we have

$$\begin{aligned}
\alpha_1 x_1 + \beta_1 x_2 &= 0, \\
\beta_1 x_1 + \alpha_2 x_2 + \beta_2 x_3 &= 0, \\
&\cdots\cdots\cdots\cdots \\
\beta_{n-2} x_{n-2} + \alpha_{n-1} x_{n-1} + \beta_{n-1} x_n &= 0, \\
\beta_{n-1} x_{n-1} + \alpha_n x_n &= 0.
\end{aligned} \tag{36}$$

The usual method of solving (36) is some variant of systematic elimination, and the difficulties to be mentioned below are found somewhere in any such variant. Although perhaps not the best form of elimination, one common method of solving the system (36) is the following.

We consider first the REGULAR case, where no $\beta_i=0$. Let $x_n=1$. Solve the last equation of (36) for x_{n-1}. Using these values of x_n and x_{n-1}, solve the next-to-last equation for x_{n-2}, *etc.* Finally, solve the second equation for x_1. The first equation should be satisfied, and serves to check the solution. The solution involves division by each of the β_i in turn.

In the SINGULAR case (meaning only that certain $\beta_i=0$; the determinant of B vanishes for both our regular and singular cases) we first decompose the matrix B into the direct sum of tridiagonal submatrices for which no $\beta_i=0$. Then at least one of the submatrices has a zero determinant, and in fact Givens' eigenvalue algorithm will have told us which ones. One then applies the above regular method to any such submatrix B_1 to find an eigenvector. Finally, one gets a corresponding eigenvector of A by setting equal to zero each component of x belonging to dimensions not in B_1.

As an example of the singular case, suppose that

$$B = \begin{bmatrix}
2 & 1 & & & & \\
1 & 2 & 0 & & & \\
 & 0 & 1 & 1 & & \\
 & & 1 & 1 & 0 & \\
 & & & 0 & 2 & 1 \\
 & & & & 1 & 2
\end{bmatrix}. \tag{37}$$

Since the middle block ($i=3, 4$) has zero determinant, we solve it and find $x_4=1$, $x_3=-1$. Then the corresponding eigenvector of B is

$$x = (0, 0, -1, 1, 0, 0). \tag{38}$$

We thus have algorithms for both the regular case (no $\beta_i=0$) and the singular case (some $\beta_i=0$), and they are distinct. The thesis of this paper is that the boundary between these cases cannot be a sharp one. The question here takes the form: when should a small number be considered zero for practical purposes?

If some of the β_i are nearly zero, the regular method involves division by nearly-zero numbers, and Givens' experience (oral communication) is that often the resulting machine solution of (36) turns out to yield meaningless numbers. To illustrate the difficulty in a simple case, suppose that we alter B in (37) to read

$$B = \begin{bmatrix} 2 & 1 & & & & \\ 1 & 2 & \epsilon & & & \\ & \epsilon & 1 & 1 & & \\ & & 1 & 1 & \epsilon & \\ & & & \epsilon & 2 & 1 \\ & & & & 1 & 2 \end{bmatrix}. \tag{39}$$

If ϵ is very close to 0, the determinant of B is practically 0, and there is an almost-zero eigenvalue of B. In solving the system (36) with the coefficients of (39), the divisions by ϵ cause an excessive magnification of the round-off error. Suppose, for example, that $\epsilon=0.0001$ and that we work to 6 significant figures with all numbers "rounded down" to a smaller absolute value. (We use downward rounding in this simple example to obtain some round-off error and thus to simulate the effect of ordinary round-off in more realistic examples.) One eigenvalue of B turns out to be 0 to more than 7 decimal places. To find the corresponding eigenvector x, we solve (36) recursively and find

$$\begin{aligned} x_6 &= 1.00000, & x_4 &= 30000.0, & x_2 &= -1000.00, \\ x_5 &= -2.00000, & x_3 &= -29999.9, & x_1 &= 2002.99. \end{aligned} \tag{40}$$

We suspect trouble when the values x_1, x_2 fail to satisfy the first equation of (36). When the numbers in (40) are normalized, we find

$$\begin{aligned} x_1 &= .066766, & x_3 &= -.999996, & x_5 &= -.000066, \\ x_2 &= -.033333, & x_4 &= .999999, & x_6 &= .000033. \end{aligned}$$

To 6 decimals, the true eigenvector is

$$\begin{aligned} x_1 &= -.000033, & x_3 &= -1.000000, & x_5 &= -.000067, \\ x_2 &= .000067, & x_4 &= 1.000000, & x_6 &= .000033. \end{aligned} \tag{41}$$

We thus see that for the nearly-singular case $\epsilon=.0001$, the values of x_1 and

x_2 are obtained far more accurately from (38) by putting $\epsilon=0$ than by attempting to apply the ordinary method of solving (36) with limited precision. This is a good example of method (i) of the Introduction.

For a more accurate determination of x in the almost-singular case we must study the singular method to see how to recast the solution of (36) in the regular case. For the present problem the recast method can look like this. We first examine the three minors into which B is *almost decomposed* by the smallness of ϵ, and find that the middle minor has the determinant nearest zero. We therefore solve the third and fourth equations of (36) for x_3 and x_4 (up to a scalar multiple), ignoring the coefficients ϵ in these equations only. Using the values of x_3, x_4, we then determine x_1, x_2 from the first two complete equations (including ϵ), and x_5, x_6 from the last two. We used this method to obtain the solution (41).

In this way one makes symmetrical use of all 6 equations, in contrast with our earlier method. The solution no longer consists of simple recurrences, however, and each block has to be solved as a simultaneous equation system with a tridiagonal matrix. But such systems are comparatively easy to solve in various ways.

If two or more blocks of (39) had both been nearly singular, enough error might have resulted from dropping the ϵ in the third and fourth equations to have warranted an iteration of the above process. That is, having found an approximate eigenvector x, we might have used the values of x_2 and x_5 with the previously ignored ϵ to modify the solution x_3, x_4 of the third and fourth equation obtained previously. With these improved values of x_3, x_4 one would once again have computed x_1, x_2 and x_5, x_6.

The author believes that a development of the above ideas might permit a more accurate solution of any nearly-singular tridiagonal system (36) than is possible by just replacing all small numbers by 0. The important idea is to look at the nearly singular ($\epsilon\approx 0$) in the light of the truly singular ($\epsilon=0$).

Givens has recommended the solution of systems (36) by triangularization with orthogonal transformations of the rows. See Dykaar and La Budde [2].

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UNDERGRADUATE RESEARCH IN MATHEMATICS

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A creative mathematician is the intersection of several unlikely events. For the most part we are ignorant of the nature of these events and of their probabilities. Some of them are at present quite beyond our control. An example is the probability of a genetic composition necessary for intensive and highly abstract thinking. Others are clearly subject to our influence. For example, the probability of an early acquaintance with living mathematics and with the joy of mathematical achievement is determined by educational practices. This article describes a program designed to increase this probability by providing unusually able undergraduates with appropriate mathematical experiences.

1. The program in outline. Selected freshmen are invited to enroll in an honors section in which they may remain for their first two years. Students are encouraged to begin original mathematical investigations either independently or in collaboration with other students and faculty members. During the junior and senior years research assistantships are available. A weekly faculty-student colloquium serves as a focus for activity at this level.

2. The context. Carleton College is a coeducational liberal arts college with less than 1,000 students. There is no college requirement in mathematics, but more than half the freshman class takes the elementary course, about half of these take more mathematics, and a good proportion of the most able students major in science. Intellectual interests are respectable, and there is considerable extra-curricular activity in mathematics.*

3. Selection and recruiting. During the summer, folders for incoming freshmen are studied with a view to identifying students with unusual promise in mathematics. Scores in *both* verbal and mathematical aptitude tests of the College Entrance Examination Board have served as one of the most useful criteria. A small group of students with fairly homogeneous background and ability is selected. Letters of invitation are sent so that incoming freshmen receive them when they arrive on campus. These letters explain the nature of the honors section. They make clear that the invitation is independent of the student's intended major and that he may drop out at any time without penalty. The number of students invited has been about ten percent of those taking freshman mathematics. About ten percent of these have declined, usually on the grounds that we have overestimated their mathematical abilities. A very few uninvited students request permission to register for the section. These include students on the borderline as judged by their records, upperclassmen who wish to take

* For further details on the program at Carleton see Report on a Conference on Undergraduate Mathematics Curricula held at Hunter College, New York City, October 12–13, 1956. This report has been circulated to mathematics departments, and additional copies may be obtained from the National Science Foundation, Washington 25, D.C.

elementary mathematics in a more stimulating class, and an occasional "crasher" who grossly overestimates his mathematical ability. It has proved satisfactory to admit a few students in the first two categories, but it is necessary to firmly exclude those in the third.

Research assistants are chosen informally by the entire staff on the basis of demonstrated interest and ability. Assistantships are awarded to nonmajors with the necessary interest and qualifications. We try to draw the student into research projects in a natural way, so that the appointment as a research assistant is a recognition of an already existing situation.

4. Grading. Students in the honors sections are graded in comparison with all the students taking elementary courses. This is achieved in part by including common questions on the examinations and in part by comparing the students with previous college generations in terms of their performance in attacking unusually difficult problems. The average grade in an honors section is expected to be somewhere between an A and a B, usually with more A's than B's being assigned. Students who do not maintain this standard are asked to transfer to another section. They may also transfer at their own request. These two categories have amounted to about 20 percent during or at the end of the freshman year.

5. Subject matter and texts. The usual freshman-sophomore sequence in mathematics consists of an introduction to logic, set theory, and other notions fundamental to modern mathematics, followed by a course in the calculus through manipulative differential equations and the calculus of several variables. The honors sections cover this material with considerable enrichment in three semesters instead of four. The second semester of the sophomore year is devoted to projects chosen by the class. The honors sections use the same textbooks as the regular sections and supplement them by outside reading. We found by experiment that students preferred to use the same textbooks in order to be assured that they were not missing any topic being covered by their classmates in the other sections.

6. Teaching procedures. Experience indicates the importance of keeping in mind that the honors students are unusually able and otherwise very much like their classmates. They have become accustomed in high school to being very obviously the outstanding students in a class. The experience of finding themselves to be merely one of many able students is sometimes traumatic. Most of them have not been accustomed to working hard in order to get top grades, and they may react negatively to the new challenge and the heavy competition. Typically, they are hard-working, careful, accurate, ambitious, and well-adjusted young people. They need to be challenged, but they also need plenty of encouragement and individual attention.

We found it undesirable to spend much class time on routine matters or on drill. Students of this capacity quickly mastered definitions and techniques. It

proved satisfactory to assign a moderate number of graded problems to be done by the student and kept in a loose-leaf notebook which was inspected occasionally. Frequent quizzes served to assure both the students and the teacher that routine matters were being mastered.

In order to encourage independent choice of problems, students were told that they would be *permitted* to hand in one problem a week of their own choice. This might be from the part of the book currently being discussed, from previous or future problem sets, or from any other source. In any case, a student was to select the problem which in his opinion represented his best work for the week. It was pointed out to the class that this manner of judging people was similar to the way in which scientists were judged by their colleagues, namely in terms of their best work. This device was very successful in stimulating work and in encouraging good exposition.

The class was conducted on an informal basis. Some time was taken for lectures introducing supplementary subject matter, but most of the class time was devoted to student exposition of their best work, to discussion of difficult points, and to cooperative effort in attacking difficult problems. Students were expected to meet their teacher frequently outside the class, either singly or in groups.

Examinations typically consisted of a few routine problems followed by a choice among quite difficult and unfamiliar ones. It was found desirable not to give too many choices, because of the time wasted trying to choose among many possibilities. We found that students of this ability could be expected to construct original proofs of unfamiliar statements of the level of difficulty of typical epsilon-delta proofs and of such theorems as the extended law of the mean.

Undoubtedly the most important thing in teaching the honors sections is to take the time to think about the individuals in the class and to work out individual measures to help them develop. This implies, of course, small classes and adequate time outside of class for meditation and student conferences. Probably fifteen is an upper bound to the size of a class in which one can do effective work of this kind.

7. Research projects. Freshmen and sophomores may be encouraged to think of mathematics in terms of creative work by giving them opportunities to develop some class topics on their own. For example, the theory of simplifying the general equation of a conic section by rotation and translation was given as a problem on a final examination. Another example is a term paper in which students were asked to develop the formula for the distance between a point and a straight line by several different methods. Sometimes students may become intrigued by an unsolved problem mentioned by the professor and, if given encouragement, may work on the problem over a long period of time. By mentioning research activities being carried on in the department, the teacher may provide the spark that starts the student studying and thinking about mathematics on his own. Once the student has taken this step, his mathematical

development accelerates sharply. Indeed, the taking of this step marks the transition toward which the program is aimed. The student who has begun to work on his own in this way, rather than as a mere doer of class assignments, still needs encouragement and guidance, but he is now very likely to develop his full capabilities in science. We know of students who continued to work during the entire undergraduate period on a problem presented to them in a freshman class, and who learned a great deal of mathematics in order to be able to attack the problem more effectively.

In addition to the casual mention of current research activities, the department has issued brief descriptions of current research projects and encouraged students to discuss them with the faculty members and students involved. A research assistant who completes a project or who is appointed without having any specific project in advance is expected to work out a program in consultation with members of the staff. His activities may vary from solving difficult problems, such as those given in the Putnam Competition, to carrying on a long-term project. We try to help the student choose problems suitable to his abilities and interests. We are not concerned as to whether results will be publishable. The student who obtains previously published results by doing work that is original for him, gains as valuable an experience as if the work had never been done before. Indeed, the fact that his results have previously been published is often very encouraging to the undergraduate because it assures him that his work is of a quality worthy of publication.

8. The colloquium. Approximately once a week the mathematics staff and research assistants meet informally in a room with comfortable chairs and a blackboard. The programs at the colloquia are varied: an introduction to some research problem, a report on research activities, an expository talk by a visiting mathematician, a joint discussion of a paper or book that has been read by the group, or a cooperative attack on a problem. Participation in the colloquium is open to all interested students and faculty at Carleton College and at neighboring St. Olaf College.

9. Examples of research activity. In a freshman class, the professor presented the famous Tower of Hanoi problem as an example of proof by induction. The number of moves required to transfer the discs from one of three needles to another is well known. The professor dropped the remark that very little was known about the cases in which the number of needles was greater than three. A girl in the class began working on this more general problem. She obtained some results by experimentation and continued to work on the problem off and on during the next year. In her junior year she made further progress on the project by making use of the calculus of finite differences. After her results were presented to the colloquium other students and faculty became interested and further results were obtained. The problem has not been completely solved to date.

Some sophomores visited the colloquium and decided that it would be a nice idea to organize a group of their own during the first semester of the junior year. This group met together and decided to study the calculus of finite differences. They met regularly and worked their way through an elementary textbook. No faculty member ever met with them, and they were not examined or given any credit. However, it was very evident that they had learned a great deal because they later frequently used finite difference methods in attacking problems.

After solving the familiar problem of the sailors on a desert island who divide up a pile of coconuts, one student conceived the idea of generalizing the problem to an arbitrary number of sailors and an arbitrary sequence of discards before each division of the pile. By using difference equations and some ideas from number theory he worked out an algorithm for solving any problem of this kind. His formulation of the problem and his method of attack were conceived without any faculty assistance. A faculty member helped him to polish up some details of the solution. He presented his results at a state meeting of the Mathematical Association.

In a course in number theory, the professor wrote every day on the side board some difficult problems which the students might attack if they became bored. Students who did such problems were invited to present their results to the class. One such presentation inspired a freshman student to generalize the problem and discover new results. His work stimulated a sophomore to discover that the number of representations of an integer as a sum of consecutive integers is equal to the number of odd divisors not including one. The professor collaborated with the students in polishing up the proofs of these results. The later discovery that the result had been published about thirty years ago, far from discouraging the students, confirmed the significance of their original work.

A professor had to calculate some rather large coefficients in connection with some work on the theory of equations. One of the research assistants did these computations for him.

A major in chemistry made a study of the mathematics and chemistry curricula in order to see how more applications to chemistry could be introduced in the mathematics courses and how more mathematics could be utilized effectively in teaching chemistry.

In describing research being carried on in the field of directed graphs, it was mentioned that the only treatise on the subject was in German. One of the research assistants conceived the idea of translating this treatise into English. In carrying out such a project he would, of course, have to learn a good deal about the subject. Another student became intrigued by some of the problems of graph theory and entered into collaboration with a faculty member working in this field.

Whenever students found themselves without a research project or momentarily against a blank wall in a project they had undertaken, it was suggested

that they attempt to solve Putnam Problems. A discussion of solutions in the colloquium usually proved stimulating. Satisfactory solutions were added to a collection of problems and solutions in the departmental library.

10. Staff and finance. A program of this kind is costly in time, since staff members must have time to think about their students, to confer with them, and to undertake the original work involved. If the honors sections are small, those teaching them do not need any special reduction in teaching load. However it is important that faculty members who are working with advanced students have a lighter teaching schedule. The program at Carleton is a four-year experiment, financed by a grant from the National Science Foundation. This grant permits the addition to the staff of one full-time person. This permits two extra sections of freshman and sophomore mathematics and a reduction of teaching load equal to one half a full-time load, which is distributed among the entire staff, since all staff members are expected to participate in the program. We have found that the size of the stipends to research assistants is not very important, because students of this calibre usually already have enough financial aid. We anticipated one or two research assistants per year and allowed about $500.00 per assistant per year. However, in the first year we had seven research assistants and gave them nominal stipends of $100.00 each.

Since research assistants are busy with a regular schedule of classes during the year and do not have ample time to pursue their mathematical interests, it would be desirable to arrange for student-faculty collaboration during the summer months. This was not provided for in our original plans, but we are trying to arrange for it.

As an alternative or supplement to reduced teaching loads, time for work with undergraduates may be made available by relieving faculty members of nonacademic chores. For this purpose adequate secretarial help is desirable.

11. Conclusions. Since the program is only in its second year of operation it is too early to draw conclusions with any certainty. We hope to follow these students' careers as undergraduates and in the future. We may also gain some insight into the subtle motivational factors involved. However, even our brief experience makes it very clear that a program of this kind is very stimulating both to faculty and to students, that it accelerates remarkably mathematical development of able undergraduates, and that the dividends are likely to be large compared to the cost. Of course any program of this kind has to be adapted to the particular institution, students, and faculty. Possibly the only universally necessary features are an enthusiastic staff and adequate time to translate enthusiasm into activity appropriate to individual students.

MÖBIUS TETRADS II*

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5. Some results from Part I. We recall (Sec. 1.1) that $T_0 \equiv ABCD$ and $T_1 \equiv A_1B_1C_1D_1$ are a pair of Möbius tetrads and that AB_1, A_1B, CD_1, C_1D are a set of four skew lines intersecting another set AB, CD, A_1B_1, C_1D_1, forming two systems of generators of a quadric Q_1 that is inscribed as well as circumscribed to both T_0 and T_1, *i.e.*, T_0 and T_1 are m.i.† We denote by u and v the pair of generators of Q_1 that separate harmonically the two pairs of generators of the first system, AB_1, A_1B and CD_1, C_1D. A quadric Q may be constructed (Sec. 2.3) such that both T_0 and T_1 are self-polar for Q. Also, $T_2 \equiv A_2B_2C_2D_2$ is a third tetrad (Sec. 3.1) interlocked with both T_0 and T_1 such that A_2 lies in BCD and $B_1C_1D_1$, B_2 in ACD and $A_1C_1D_1$, C_2 in ABD and $A_1B_1D_1$, D_2 in ABC and $A_1B_1C_1$. It was shown (Sec. 3.3) that T_2 is also self-polar for Q.

6. Sets of tetrads and associated quadrics.

6.1. A fourth tetrad leading to two sets of four m.i. tetrads. Let l_1 and l_2 be lines in $B_2C_2D_2$ common to BCD and $B_1C_1D_1$, respectively; and l_1' and l_2', in $A_2C_2D_2$ common to ACD and $A_1C_1D_1$, respectively. Through the pole of AA_1A_2 for Q, say A_3, where the three planes BCD, $B_1C_1D_1$, and $B_2C_2D_2$ meet (*i.e.*, where l, l_1, and l_2 meet) draw a secant to u and v that will meet l', l_1', and l_2' in their common point where the three planes ACD, $A_1C_1D_1$, and $A_2C_2D_2$ meet, *i.e.*, the pole of BB_1B_2, say B_3. By an argument of Section 3.4 according to which A_3B_3 meets CC_1, DD_1, C_1C_2, D_1D_2, C_2C, and D_2D, it follows that A_3B_3 is the common line of the planes CC_1C_2 and DD_1D_2. Similarly, C_3D_3 is the common line of the planes AA_1A_2, BB_1B_2, and secant to u and v, where C_3 is the pole of CC_1C_2, and D_3, of DD_1D_2 for Q. Now, A_3, B_3 and C_3, D_3 as located are pairs of conjugate points for Q, proving that $A_3B_3C_3D_3$ is a tetrad T_3, self-polar for Q and interlocked (Secs. 3.3, 3.4) with T_i $(i=0, 1, 2)$ such that T_i $(i=0, 1, 2, 3)$ *form a set of four m.i. tetrads giving rise to another set of four m.i. tetrads, viz.*, $T_A = AA_1A_2A_3$, $T_B = BB_1B_2B_3$, $T_C = CC_1C_2C_3$, $T_D = DD_1D_2D_3$, *having the same vertices and faces as those of the first set.*

The two sets may be called conjugate to each other with the property that each tetrad of either set has one vertex and its opposite face common with each tetrad of the other set. The eight tetrads of the two sets are evidently self-polar for Q.

6.2. Observations. (a) We now observe that the sixteen vertices of the four tetrads of the preceding section lie in the sixteen faces of the same tetrads by sixes that form two self-conjugate triangles for Q, *i.e.*, for the conic section of Q by the face in which they lie. For example, A_1, A_2, A_3, and B, C, D, lie in a face and form self-polar triads for Q. Here we are reminded of a well-known property

* Part I of this paper appeared in this MONTHLY, vol. 64, 1957, pp. 471–478. The sections of Part II are numbered to follow those of the first.

† m.i. denotes mutually interlocked.

(**B-2**, p. 35, ex. 12)* of a pair of self-conjugate triangles for a conic, that is, their vertices will lie on a conic and their sides will touch a conic. Hence, we may infer that *the sixteen vertices of four m.i. tetrads lie by sixes on sixteen conics, one in each face of the same tetrads, having a pair of vertices common with one another* (**B-4**, p. 131). We shall speak of *the sixteen conics as associated with the two conjugate sets of four m.i. tetrads.*

(b) We also observe (reciprocally) that *the sixteen faces of four m.i. tetrads pass through, by sixes, sixteen vertices of the same tetrads and envelope, by sixes, sixteen cones whose vertices are the vertices of these tetrads having a pair of tangent faces common with one another.* We shall speak of *the sixteen cones as associated with the two conjugate sets of four m.i. tetrads.*

This reciprocation with respect to the quadric Q for which the four m.i. tetrads are self-polar brings us back to the same set of tetrads. We may, however, reciprocate with respect to any quadric whatever and observe that the reciprocal set of four tetrads will also be m.i. and that what we have said (reciprocally) above will also hold for this set of four m.i. tetrads. Moreover, whatever property holds for a set of four m.i. tetrads must also hold for any set of four such tetrads.

6.3. Associated quadrics. (a) Let a quadric Q_{1AB} be constructed to pass through the four vertices of T_0, three of T_1, and two, A_2, B_2, of T_2, where T_0, T_1, and T_2 belong to a set of four m.i. tetrads (Sec. 6.1). Since Q_{1AB} is out-polar (**B-3**, p. 52) to Q, it will pass through the fourth vertex of T_1 and two vertices, A_3, B_3, of the fourth tetrad of the set, these being the fourth vertices of the two tetrads T_A and T_B of the conjugate set, self-polar for Q; the other vertices of T_A and T_B are already there. Hence Q_{1AB} passes through four of the sixteen conics of the preceding section *viz.*, $BCDA_1A_2A_3$, $B_1C_1D_1AA_2A_3$, $ACDB_1B_2B_3$, $A_1C_1D_1BB_2B_3$.

We can choose the first two tetrads of a set in six ways and two vertices of a third (implying two vertices of the fourth also) in six ways, proving that *there are thirty-six quadrics out-polar to Q, each passing through four of the sixteen associated conics and twelve vertices of two pairs of tetrads (e.g., T_0, T_1 and T_A, T_B as above), one pair from each of the two conjugate sets of four m.i. tetrads each.*

The symmetry of the result shows that we may choose either set and still have the same thirty-six quadrics. A little calculation shows that nine quadrics of the thirty-six pass through each conic.

(b) We may also state a reciprocal result in the same manner in which (b) was obtained from (a) in Section 6.2. In case reciprocation is made with respect to the quadric Q, the tetrads under consideration reciprocate into themselves, being self-polar for Q. Hence, *the latter thirty-six quadrics will be the reciprocals of the former thirty-six with respect to Q and, therefore, will be inpolar to Q.* In case reciprocation is made with respect to any other quadric, say Q^*, then Q

* **B-i** denotes H. F. Baker, Principles of Geometry, vol. i ($i=1, 2, 3, 4$).

will be replaced in what we have just said by its reciprocal quadric, say Q', with respect to Q^*.

7. Sets of quadrics.

7.1. Enumeration. We made a passing reference at the end of Section 1.1 that there are three quadrics of the type Q_1 for a pair of Möbius tetrads. Let us name the other two, for T_0 and T_1, as Q_1' and Q_1'' generated by BC, AD, B_1C_1, A_1D_1 and CA, BD, C_1A_1, B_1D_1, respectively, Q_1 being the one already defined (Sec. 1.3). *For a set of four m.i. tetrads T_i ($i=0, 1, 2, 3$), there will therefore be eighteen such quadrics, three for each pair of tetrads and all self-reciprocal for Q* (Secs. 2.3, 2.5). We designate the sets of three quadrics as Q_2, Q_2', Q_2'' ; Q_3, Q_3', Q_3'' ; $Q_{12}, Q_{12}', Q_{12}''$; $Q_{23}, Q_{23}', Q_{23}''$; $Q_{31}, Q_{31}', Q_{31}''$; for the five pairs of tetrads T_0, T_2; T_0, T_3; T_1, T_2; T_2, T_3; T_3, T_1; respectively, in the same manner as Q_1, Q_1', Q_1'' were defined above for the sixth pair T_0, T_1.

(b) *These eighteen quadrics of the preceding paragraph may be divided into three sets of six each, viz., $Q_{ij}, Q_{ij}', Q_{ij}''$, where $i, j=1, 2, 3$, $ij=ji$ for $i\neq j$, and $ij=ji=i$ for $i=j$. Each set of six quadrics has a pair of generators of the same system in common with Q.*

It may be noted here that *these three pairs of generators separate each other harmonically and the three couples of the pairs of four m.i. tetrads of the conjugate set* (*Sec.* 6.1), *viz.*,

(1) T_A, T_B; T_C, T_D; (2) T_B, T_C; T_A, T_D; (3) T_C, T_A; T_B, T_D;

are harmonically inverse in pairs with respect to them, respectively (**B-4**, p. 133).

This follows from the fact that if A inverts harmonically (with respect to a pair of generators of a quadric) into B, if B inverts (with respect to another pair of generators of the same system) into C and if C inverts (with respect to a third pair of generators of the same system) into A, then the three pairs of generators separate each other harmonically. For the three pairs of points on the conic section in the plane ABC of the quadric, as intersections of the plane with the above three pairs of generators of the quadric, separate each other harmonically by reason of the behavior of the three pairs of intersections of a conic with the sides of a triangle self-conjugate for it (**B-2**, pp. 27–28, ex. 5).

(c) *For the conjugate set of m.i. tetrads T_i ($i=A, B, C, D$) we shall similarly have eighteen other quadrics, three for each pair of tetrads, all self-reciprocal for Q.* In this case, the quadrics may be designated by $Q_{ij}, Q_{ij}', Q_{ij}''$, where $i, j=A, B, C, D$, $i\neq j$, $ij=ji$.

7.2. Further relations. (a) We may observe from their definitions of the quadrics of a set Q_{ij}, Q_{ij}', or Q_{ij}'' of the preceding section that the three of them circumscribing the same tetrad form a pencil of quadrics having two pairs of generators in common and that there are four such pencils.

Hence, if each quadric be represented by a point, the quadrics of the set can be represented by the vertices of a quadrilateral assuming the pencil of quadrics

equivalent to a range of collinear points, the relation being linear in either case.

(b) We notice further that the three diagonal points of each quadrilateral will lead to three distinct quadrics which confirm the harmonic property of the quadrilateral. This is clear from consideration of the nine new quadrics represented by the nine diagonal points of the three quadrilaterals representing the three sets of quadrics Q_{ij}, Q'_{ij}, Q''_{ij}, for either of the two conjugate sets of four m.i. tetrads each. Let us designate the quadrics represented by the diagonal points where the pairs of lines joining the opposite vertices Q_{ij}, Q_k; Q_{ki}, Q_j meet by $Q_{i,jk}$, $i, j, k = 1, 2, 3$, $ij = ji$, $ki = ik$, $jk = kj$, $i \neq j \neq k$; where Q_{ij}, Q_{kl}; Q_{jk}, Q_{li} meet by $Q_{ki,jl}$, $i, j, k, l = A, B, C, D$, $ij = ji$, $kl = lk$, $jk = kj$, $li = il$, $ki = ik$, $jl = lj$, $i \neq j \neq k \neq l$.

We should like to note here that *the nine quadrics* $Q_{i,jk}$, $Q'_{i,jk}$, $Q''_{i,jk}$ *representing the diagonal points of the three quadrilaterals* Q_{ij}, Q'_{ij}, Q''_{ij} *arising out of a set of four m.i. tetrads are the same as those arising out of the conjugate set, viz.,* $Q_{ij,kl}$, $Q'_{ij,kl}$, $Q''_{ij,kl}$. *The order is such that the three planes represented by the quadrilaterals* Q_{ij}, Q'_{ij}, Q''_{ij} *of either set, being skew to each other, have each one point in common with each of the three such planes representing the quadrics of the conjugate set. Each plane consists of nine points and seven lines only, viz., the six vertices, the three diagonal points, the four sides, and the three diagonals of the quadrilateral it represents. These nine new quadrics are observed to have two pairs of generators, each common with Q, one pair from either of the three pairs of generators of the same system associated with the eighteen quadrics* (Sec. 7.1) *related to either of the two conjugate sets of four m.i. tetrads each* (*Fig. 5*).

Quadric Q

$Q_{1,23}$	(1) $\equiv$	$Q_{AB,CD}$; $Q'_{1,23}$	(1′) $\equiv$	$Q_{BC,AD}$; $Q''_{1,23}$	(1″) $\equiv$	$Q_{CA,BD}$
$Q_{2,31}$	(2) $\equiv$	$Q'_{AB,CD}$; $Q'_{2,31}$	(2′) $\equiv$	$Q'_{BC\,AD}$; $Q''_{2,31}$	(2″) $\equiv$	$Q'_{CA,BD}$
$Q_{3,12}$	(3) $\equiv$	$Q''_{AB,CD}$; $Q'_{3,12}$	(3′) $\equiv$	$Q''_{BC,AD}$; $Q''_{3,12}$	(3″) $\equiv$	$Q''_{CA,BD}$

FIG. 5

(c) Again, we observe that *the thirty-six quadrics associated with the two conjugate sets of four m.i. tetrads* (Sec. 6.3) *also arrange themselves* (*on their representation by points*) *into two sets of six planes each. Each plane is determined by a quadrilateral of six quadrics represented by its vertices such that the planes of a set are skew to each other and have each point common with each plane of the other set, the sets of planes being formed according as we count the quadrics passing through the six pairs of tetrads of either of the two conjugate sets of four m.i. tetrads.*

(d) To be more explicit we may put down the scheme of the two sets of six planes as Q_{1ij}, Q_{2ij}, Q_{3ij}, Q_{12ij}, Q_{23ij}, Q_{31ij}, where $i, j = A, B, C, D$, $i \neq j$, $ij = ji$.

Each subscript ij represents the six quadrics of a plane of one set and each numeral subscript represents those of a plane of the second set for the six values of ij. Figure 6 illustrates the scheme of a plane of either set.

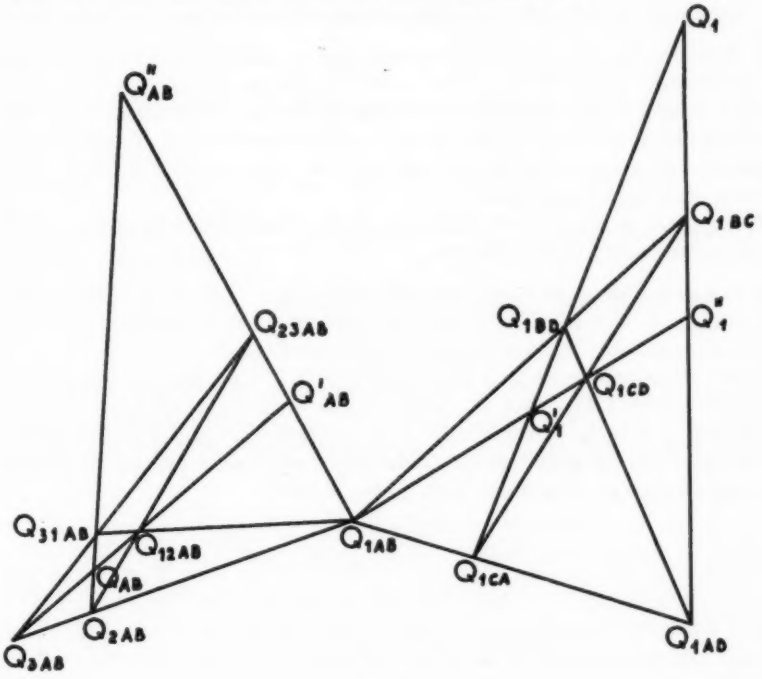

FIG. 6

The diagonal points of the six quadrilaterals of either set are observed to represent the eighteen quadrics Q_{ij}, Q'_{ij}, Q''_{ij}, for the corresponding set of four m.i. tetrads. This suggests that we may add to each set of six planes, three more planes, viz., those determined by $Q_{i,jk}$, $Q'_{i,jk}$, $Q''_{i,jk}$, respectively, to one set and those determined by $Q_{ij,kl}$, $Q'_{ij,kl}$, $Q''_{ij,kl}$, to the other. We then have two sets of nine planes such that the planes of each set, being skew to each other, have each one point common with each plane of the other set.

For the net of quadrics Q_1, Q'_1, Q''_1 of Section 7.1 and the four pairs of Möbius tetrads of Section 1.2, reference should be made to two papers by W. L. Edge.*

My thanks are due to the referee for the present form of the paper and to Professor B. R. Seth for his continual encouragement to carry on with my research.

* The net of quadric surfaces associated with a pair of Möbius tetrads, Proc. London Math. Soc. (2), vol. 41, 1936, pp. 337–360; The contact net of quadrics, *ibid.*, vol. 48, 1943, pp. 112–121.

ON A THEOREM OF HÖLDER

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1. Introduction. A well-known result, due to Hölder [1], is the following: The symmetric group S_n has outer automorphisms if and only if $n=6$. The classical proof of the existence of a class of outer automorphisms of S_6, as formulated by Burnside [2], rests in part on the theory of primitive groups and entails extensive computation. In this note we offer a direct method for constructing such automorphisms.

The author is grateful to Professor R. H. Bruck for raising this problem and for subsequent helpful remarks.

2. Construction of an outer automorphism of S_6. Let S_6 be defined on the set $M=\{1, 2, 3, 4, 5, 6\}$; let I denote the identity of S_6. Call two elements of S_6 *disjoint* if no element of M is displaced by both of them.

Define the mapping ψ by: $(1\ 2)\psi=(1\ 2)(3\ 6)(4\ 5)=P_2$, $(1\ 3)\psi=(1\ 3)(2\ 4)(5\ 6)=P_3$, $(1\ 4)\psi=(1\ 4)(2\ 6)(3\ 5)=P_4$, $(1\ 5)\psi=(1\ 5)(2\ 3)(4\ 6)=P_5$, $(1\ 6)\psi=(1\ 6)(2\ 5)(3\ 4)=P_6$. Write $N=\{2, 3, 4, 5, 6\}$, $\mathcal{P}=\{P_i \mid i\in N\}$. Note that the elements of $\mathcal{P}$ include as factors the 15 distinct transpositions of S_6; consequently $\mathcal{P}$ is transitive on M. Moreover, for $i, j, k\in M$, $i\neq j$,

$$P_i^2 = I, \qquad kP_i \neq kP_j, \qquad iP_j \neq i.$$

Note that $iP_j=jP_i$ implies $i=j$. For if $iP_j=jP_i=k$ then $P_i=(1\ i)(j\ k)(r\ s)$, $P_j=(1\ j)(i\ k)(r\ s)$, so $i=j$. Also, $P_iP_j=(i\ \ j\ \ jP_iP_j\cdots(1\ \ iP_j\ \ iP_jP_iP_j\cdots$. Hence $(jP_iP_j)P_iP_j$ equals i or 1. But in the latter case $jP_iP_j=1P_jP_i=jP_i$, whereas P_j fixes no element of M. Thus P_iP_j has order three, so $P_iP_jP_i=P_jP_iP_j$, all $i, j\in N$.

If i, j, k are distinct elements of N, then

$$iP_j = jP_k = kP_i \tag{1}$$

cannot hold. For, if so, write $iP_j=q$ and $N=\{i, j, k, q, r\}$. Now $q=fP_r$ for some f in M. Certainly f is not one of i, j, k, or q. But if $f=r$ then $q=rP_r=1$, contradicting $i\neq j$.

If P_i, P_j, P_k are distinct elements of $\mathcal{P}$, then

$$(P_iP_kP_j)P_i = P_j(P_iP_kP_j). \tag{2}$$

It is sufficient to prove that P_k commutes with $P_iP_jP_i$, for then $P_kP_iP_jP_i=P_iP_jP_iP_k$, $P_iP_kP_iP_jP_i=P_jP_iP_k$, $P_iP_kP_jP_iP_j=P_jP_iP_kP_jP_j$, $P_iP_kP_jP_i=P_jP_iP_kP_j$. Now

$$Q = P_iP_jP_i = P_jP_iP_j = (1\ \ iP_jP_i)(i\ \ jP_i)(j\ \ iP_j).$$

Each of the three transpositions of Q is a factor of some P_k, $k\neq i, j$. If Q should have two cycles in common with some P_t then $Q=P_t$. But in that case the dis-

played representation of Q would yield $iP_j=jP_t$, $iP_jP_i=t$ (so $iP_j=tP_i$), whence $tP_i=iP_j=jP_t$, contradicting (1). (Thus we can write $Q=(a\ b)(c\ d)(e\ f)$, $P_k=(a\ b)(c\ f)(d\ e)$. But then $QP_k=(c\ e)(d\ f)=P_kQ$.

If $A_1, \cdots, A_n, B, C$ are distinct elements of $\mathfrak{P}$, then

$$B(CA_1 \cdots A_nB) = (CA_1 \cdots A_nB)C. \tag{3}$$

If $n=1$, (3) follows from (2). Assume inductively that (3) holds for n; then

$$\begin{aligned} &B(CA_1 \cdots A_nA_{n+1}B) \\ &= B(CA_1 \cdots A_nB)(BA_{n+1}B) \quad = \quad (CA_1 \cdots A_nBC)(A_{n+1}BA_{n+1}) \\ &= (CA_1 \cdots A_nA_{n+1})(A_{n+1}BCA_{n+1})BA_{n+1} = (CA_1 \cdots A_nA_{n+1})(BCA_{n+1}B)BA_{n+1} \\ &= (CA_1 \cdots A_nA_{n+1}B)C. \end{aligned}$$

Further, if $A_1, \cdots, A_n, B, C$ are distinct elements of $\mathfrak{P}$, then

$$CB(A_1 \cdots A_n)B = B(A_1 \cdots A_n)BC. \tag{4}$$

For by (3), $CBA_1 \cdots A_nB = BC(BCA_1 \cdots A_nB) = BC(CA_1 \cdots A_nBC) = B(A_1 \cdots A_nBC)$.

Define the mapping θ as follows. Let $a_1, \cdots, a_n$ be distinct elements of N and write $(1\ a_i)\psi=A_i$. Then set

$$\begin{aligned} I\theta &= I, \qquad (1a_1 \cdots a_n)\theta = A_1 \cdots A_n, \\ (a_1a_2 \cdots a_n)\theta &= A_nA_1A_2 \cdots A_n, \qquad (QR)\theta = (Q\theta)(R\theta), \end{aligned} \tag{5}$$

where Q, R are arbitrary disjoint cycles of S_6. By (3),

$$(a_1a_2 \cdots a_n)\theta = A_1A_2 \cdots A_nA_1.$$

Clearly θ maps S_6 into itself.

To show that θ is single-valued it will be sufficient to establish that if $Q=(a_1 \cdots a_m)$, $R=(b_1 \cdots b_n)$ are arbitrary disjoint cycles in S_6, then

(i) $(QR)\theta=(RQ)\theta$;
(ii) $(a_1a_2 \cdots a_m)\theta=(a_2a_3 \cdots a_ma_1)\theta$.

If Q displaces 1 then $Q\theta$ is uniquely defined; if not, (ii) follows from (3). As to (i), suppose without loss of generality that R does not displace 1; then $R\theta$ is of the form $BA_1 \cdots A_nB$, so by successive applications of (4), $(QR)\theta=(Q\theta)(R\theta)=(R\theta)(Q\theta)=(RQ)\theta$.

For arbitrary elements Q, R of S_6, $(QR)\theta=(Q\theta)(R\theta)$. To prove this it is sufficient to consider the case where R is a transposition (since every element of S_6 is a product of transpositions). If Q and R are disjoint the asserted relation is trivial. Hence we write Q as a product of disjoint cycles and let Q' denote the product of those factors of Q which are not disjoint from R. We need to show that $(Q'R)\theta=(Q'\theta)(R\theta)$.

Let $1, e, f, a_1, \cdots, a_m, b_1, \cdots, b_n$ denote distinct elements of M.

(i) If $Q' = (1\ a_1 \cdots a_m)$, $R = (1\ b_1)$, then $(Q'\theta)(R\theta) = A_1 \cdots A_m B_1 = (1\ a_1 \cdots a_m\ b_1)\theta = (Q'R)\theta$.

(ii) If $Q' = (e\ a_1 \cdots a_m)$, $V = (e\ b_1 \cdots b_n)$, then $(Q'\theta)(V\theta) = (EA_1 \cdots A_m E) \cdot (EB_1 \cdots B_n E) = EA_1 \cdots A_m B_1 \cdots B_n E = (e\ a_1 \cdots a_m\ b_1 \cdots b_n)\theta = (Q'V)\theta$.

(iii) If $Q' = (1\ a_1 \cdots a_m\ e\ b_1 \cdots b_n)$, $R = (1\ e)$, with $m, n \geqq 0$, then $(Q'\theta)(R\theta) = A_1 \cdots A_m(EB_1 \cdots B_n E) = A_1 \cdots A_m B_n E B_1 \cdots B_n = [(1\ a_1 \cdots a_m) \cdot (e\ b_1 \cdots b_n)]\theta = (Q'R)\theta$.

(iv) If $Q' = (1\ a_1 \cdots a_m)(e\ b_1 \cdots b_n)$, $R = (1\ e)$, then $(Q'\theta)(R\theta) = A_1 \cdots A_m E B_1 \cdots B_n E E = A_1 \cdots A_m E B_1 \cdots B_n = (1\ a_1 \cdots a_m\ e\ b_1 \cdots b_n)\theta = (Q'R)\theta$.

(v) If $Q' = (e\ a_1 \cdots a_m\ f\ b_1 \cdots b_n)$, $R = (e\ f)$, with $m, n \geqq 0$, then by (4), $(Q'\theta)(R\theta) = (EA_1 \cdots A_m F B_1 \cdots B_n E)(EFE) = (EA_1 \cdots A_m)(FB_1 \cdots B_n FE) = (EA_1 \cdots A_m)(EFB_1 \cdots B_n F) = [(e\ a_1 \cdots a_m)(f\ b_1 \cdots b_n)]\theta = (Q'R)\theta$.

(vi) If $Q' = (e\ a_1 \cdots a_m)(f\ b_1 \cdots b_n) = Q_1' Q_2'$, $R = (e\ f)$, then, by (ii), $(Q'\theta)(R\theta) = (Q_1'\theta)(Q_2'\theta)(R\theta) = (Q_1' Q_2' R)\theta = (Q'R)\theta$.

θ is an automorphism of S_6. Indeed, the kernel, K, of θ is a normal subgroup of S_6, so K is one of S_6, A_6, $\{I\}$, where A_6 denotes the alternating group of degree 6. But $[(3\ 6)(4\ 5)]\theta = (3\ 6)(4\ 5)$, so $K \neq S_6$, $K \neq A_6$. Therefore $K = \{I\}$ so θ is 1-1 and hence an automorphism.

Finally, θ is outer since $(1\ 3\ 5)\theta = (1\ 2\ 6)(3\ 5\ 4)$, whereas if θ were inner it would map every conjugate class of S_6 onto itself. This completes the proof.

We observe in conclusion that *all* outer automorphisms of S_6 are obtainable with the aid of the above construction. Indeed, as shown by Hölder [1], the automorphism group of S_6 has order $1440 = 2(6!)$; thus the group, $\mathfrak{I}$, of inner automorphisms is of index 2 in the full automorphism group. Hence if θ is any outer automorphism of S_6 then the right coset $\mathfrak{I}\theta$ includes all outer automorphisms of S_6.

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THE MOTION OF AN ELECTRIC BELL

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1. Introduction. One of the oldest electromechanical devices is the electric bell; however, very little reference is made to its mathematics. The simplicity of the analysis involved in determining the motion of the bell's clapper provides an easily understood example of the use of the phase-plane in solving nonlinear ordinary differential equations. Emphasis is placed on the exposition of phase-plane techniques to students of calculus, differential equations and physics.

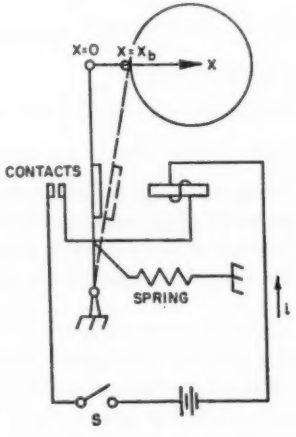

FIG. 1

The equations of motion of the bell are simplified by neglecting damping, by assuming the force due to the magnet has only two possible values, O and F, and by treating the rod carrying the clapper as being weightless and rigid, Figure 1. The equation of motion of the bell is:

$$m\ddot{x} + kx = \begin{cases} 0 & \text{when } i = 0, \\ F & \text{when } i = i_0 \neq 0 \end{cases} \tag{1}$$

where i is the current flowing through the electromagnet.

Furthermore, the restriction that the clapper cannot penetrate the body gives

$$x \leqq x_b. \tag{2}$$

The bell is designed so that the current is interrupted by a contact mounted on the clapper arm. We shall denote by x_c and x_0 the values of x for which the contacts make and break, respectively.

2. Analysis of the motion. The motion governed by (1) is best analyzed by introducing the nondimensional coordinates u and t defined by:

$$u(\tau) = kx(t)/F \qquad \tau = \omega\tau, \qquad \omega = \sqrt{k/m}. \tag{3}$$

The equation of motion now has the form:

$$u'' + u = 0, \qquad i = 0, \tag{4a}$$

$$u'' + u = 1, \qquad i = i_0, \tag{4b}$$

where $' = d/d\tau$; with the restriction

$$u \leqq u_b = kx_b/F. \tag{5}$$

Multiplying by u' and integrating gives

$$(u')^2 + u^2 = \text{const}, \qquad i = 0 \tag{6a}$$

$$u'^2 + (u - 1)^2 = \text{const}, \qquad i = i_0. \tag{6b}$$

These are equations of circles with centers at (0, 0) and (0, 1) in the u', u-plane and are called the *trajectories of the motion*, Figure 2. The u', u-plane is called the phase-plane. The radius of the circles is proportional to the kinetic energy, and the dimensionless time is measured by the arc length of the trajectory in radians.

Consider the clapper at rest, where $u' = u = 0$, and at $t = 0$, the switch S is closed completing the circuit, Figure 1. The initial motion is described by the trajectory $O-A$ in Figure 2. When the clapper reaches x_0 or $u = u_0$, the current is interrupted and the motion of the bell is described by the homogeneous equation (4a). The trajectories are now circles about the origin with the motion following the trajectory $A-B$ striking the bell. The collision must be inelastic if any sound is to be emitted and the velocity, u_c', with which the clapper leaves the bell, will be given by:

$$u_c' = -eu_b', \qquad 0 \leqq e < 1 \tag{7}$$

where u_b' is the velocity at B, and e is the coefficient of restitution. The motion continues from C to D, $u = u_c$, at which point the circuit is completed and again the motion is described by circles about the point (0, 1). The motion has completed a "cycle" when it has traveled the trajectory $OABCDO'$. A cycle can be completely described by its initial point (0, u) or the radius of the initial arc of the motion. If the motion starts at the point (0, u_m) with initial radius r_m, then, after a complete cycle, the radius has the value r_{m+1} given by:

$$r_{m+1}^2 = e^2 r_m^2 + (1 - e^2)(1 + u_b^2 - 2u_0) + 2(u_0 - u_c). \tag{8}$$

The limit cycle of the motion is defined as the cycle which the motion follows after infinite time and is characterized by $r_{m+1}/r_m \to 1$ as $m \to \infty$.*

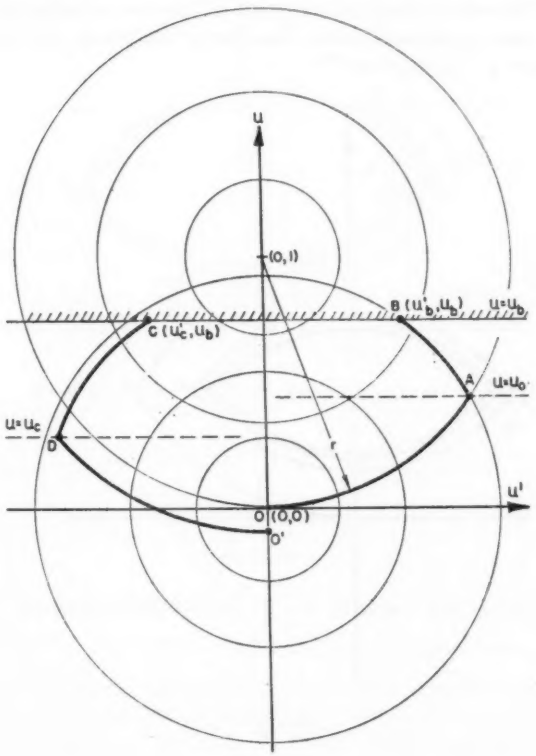

FIG. 2

In the limit cycle the radius is

$$r^2 = (1 + u_b^2 - 2u_0) + 2(u_0 - u_c)/(1 - e^2). \tag{9}$$

The motion of the bell shall now be analyzed for several special cases.

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* As the referee comments, the solution of (8) can be written

$$r_m^2 = (1 + u_b^2 - 2u_0) + 2(u_0 - u_c)/(1 - e^2) + ce^{2m},$$

which also shows the limit cycle exists.

Case I. Ideal switch on clapper arm, $u_0 = u_c$. In this case the motion "spirals" inward until the motion is bounded above by $u = u_b$ for which position $u' = 0$. This is shown graphically in Figure 3. In this case the bell starts clanging loudly, the intensity decreases, motion and the time interval between clangs is shortened. In the final state the bell does not sound, although the clapper continues to vibrate. Furthermore, energy is supplied to the system only during the O—A portion of the first cycle, and when this energy has been converted to sound, there is no longer a "ringing bell."*

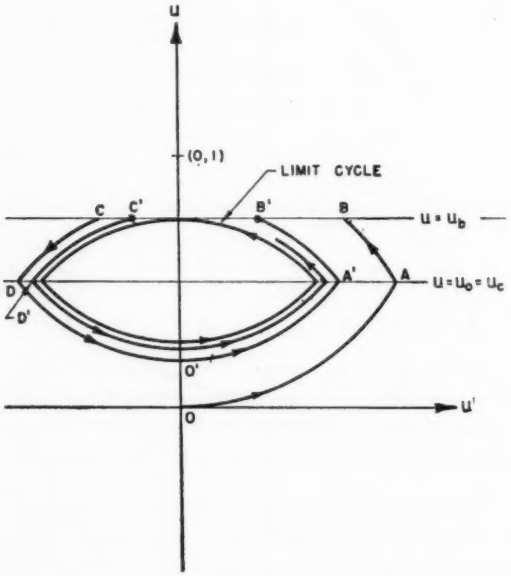

FIG. 3

Case II. Imperfect switch on clapper arm, $u_0 > u_c$. The motion of the bell when $u_0 > u_c$ yields a limit cycle as is shown in Figure 4*. Electrical energy supplied during the time interval 1, 2 of Figure 4 amounts to $E = \text{force} \times \text{distance} = u_0 - u_c$. This is precisely the energy converted into sound per cycle $E = \frac{1}{2}(u_c'^2 - u_b'^2)$ in the limit cycle.

Many more very basic physical facts can be computed from this simple example. Introduction of air damping, friction damping and nonlinear force functions, while distracting in this note, are easily treated.

* It is possible to supply insufficient energy to have any sound produced.

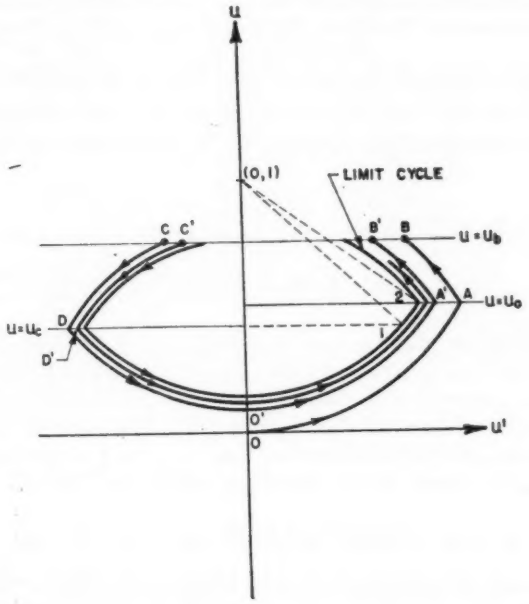

FIG. 4

NOTE ON SOME SEMIMODULI OF A RECTANGULAR MATRIX

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DEFINITION. If $\|A\|$ is a nonnegative function of the elements of a rectangular complex matrix $A=(a_{ij})$ such that (i) $\|A\|=0$ if and only if $A=0$; (ii) $\|sA\| = |s|\cdot\|A\|$ for all scalars s; (iii) $\|A+B\| \leqq \|A\|+\|B\|$ where A and B are of the same size; (iv) $\|AB\| \leqq \|A\|\cdot\|B\|$ where A and B are conformable; then we shall say that $\|A\|$ is a *semimodulus* of A.

NOTATION. Let $[N(A)]^2=\sum_i\sum_j |a_{ij}|^2$. Write $R_i(A)=\sum_s |a_{is}|$, $R(A) =\max_i R_i(A)$; $C_j(A)=\sum_s |a_{sj}|$, $C(A)=\max_j C_j(A)$. Denote by $c_{\min}(H)$ and $c_{\max}(H)$ the minimum and the maximum ch. value (read characteristic value) of the hermitian matrix H. Write $|A|_0=[c_{\max}(A^*A)]^{1/2}$, $|A|_0=[c_{\min}(A^*A)]^{1/2}$ where A^* is the conjugate transpose of A. By $c(A)$ we shall denote *any* ch. value of the (square) matrix A. For a complex number α, Re α and Im α will stand for the real and the imaginary part of α. Every square complex matrix P can be expressed as $P=P_1+iP_2$ where P_1 and P_2 are hermitian. We shall say that P_1

and P_2 are the *hermitian components* of P. This decomposition of P into its hermitian components is unique. Indeed, $P=\frac{1}{2}(P+P^*)$, $P=\frac{1}{2}i(P-P^*)$.

THEOREM 1. *$N(A)$, $R(A)$, $C(A)$ and $|A|^0$ are semimoduli of A.*

Proof. It is known that $N(A)$ satisfies (i), (ii), (iii) and (iv). So $N(A)$ is a semimodulus. Consider $R(A)$. Conditions (i), (ii) and (iii) clearly hold. To prove (iv):

$$(1)\qquad \begin{aligned} R_i(AB) &= \sum_s \Big| \sum_k a_{ik}b_{ks} \Big| \leqq \sum_k \sum_s |a_{ik}| \cdot |b_{ks}| \\ &= \sum_k |a_{ik}| \cdot R_k(B) \leqq R(B)R_i(A), \quad \text{where} \quad B=(b_{ij}). \end{aligned}$$

Since (1) is true for all i, we get $R(AB)\leqq R(A)R(B)$. Thus $R(A)$ is a semimodulus. Conditions (i), (ii) and (iii) also hold for $C(A)$. Also $C(A)=R(A')$, where A' is the transpose of A. Now $C(AB)=R(B'A')\leqq R(A')R(B')=C(A)C(B)$. So $C(A)$ is a semimodulus of A. Consider $|A|^0$. For a $(n\times n)$ hermitian matrix H and a unit-length column vector $u=\{u_1, \cdots, u_n\}$ it is known that $c_{\min}(H)\leqq u^*Hu\leqq c_{\max}(H)$. Since A^*A is hermitian we get for $|u|=1$,

$$(2)\qquad |A|_0^2 \leqq u^*A^*Au = |Au|^2 \leqq |A|^{0^2} \quad \text{or} \quad |A|_0 \leqq |Au| \leqq |A|^0.$$

Also for two column vectors $u=\{u_1, \cdots, u_n\}$ and $v=\{v_1, \cdots, v_n\}$

$$(3)\qquad |u+v| \leqq |u|+|v| \quad \text{and} \quad |u^*v| \leqq |u| \cdot |v|.$$

Further, we note that there exist unit-length ch. vectors x, y of A^*A such that the lower and upper bounds in (2) are attained, *i.e.*, $|A|_0^2=x^*A^*Ax$ and $|A|^{0^2}=y^*A^*Ay$. Now conditions (i) and (ii) hold for $|A|^0$. To prove (iii): Let x be a unit-length ch. vector of $(A+B)^*(A+B)$ such that $|A+B|^0=|(A+B)x|$. Such a vector exists as we observe above. Hence we have $|A+B|^0=|(A+B)x|=|Ax+Bx|\leqq|Ax|+|Bx|\leqq|A|^0+|B|^0$ by (2) and (3). This proves (iii). To prove (iv): Consider the unit-length ch. vector y of $(AB)^*AB$ such that $|AB|^0=|ABy|$. Two cases arise. Case (1): $By=0$. Then $|B|^0=0$. Also $ABy=0$ and so $|ABy|=0=|AB|^0$. Hence (iv) holds. Case (2): $By\neq 0$. Let $By=z=k\bar{z}$ where $\bar{z}$ is of unit length. Now $0<k=|z|=|By|\leqq|B|^0$ by (2). Hence $|AB|^0=|ABy|=|Az|=k|A\bar{z}|\leqq k|A|^0\leqq|B|^0.|A|^0$. Thus $|A|^0$ is a semimodulus and the theorem is proved.

THEOREM 2. *If X, Y, A are matrices and α a scalar such that $\alpha X=AY$ and $\|X\|=\|Y\|\neq 0$, then $|\alpha|\leqq\|A\|$.*

Proof. $\|\alpha X\|=|\alpha|\cdot\|X\|=\|AY\|\leqq\|A\|\cdot\|Y\|$. But $\|X\|=\|Y\|$ and is nonzero. So $|\alpha|\leqq\|A\|$.

COROLLARY 2.1. $|c(A)|\leqq\|A\|$.

Proof. There exists a nonnull ch. vector x such that $\alpha x=Ax$ where $\alpha=c(A)$.

Take $X=Y=x$ in Theorem 1 and observe that $\|x\| \neq 0$ and the result follows.

COROLLARY 2.2. *If* $S=\Sigma_k A_k$, $|c(S)| \leqq \Sigma_k \|A_k\|$.

Proof. $\|S\| = \|\Sigma_k A_k\| \leqq \Sigma_k \|A_k\|$ by (iii) while by Corollary 2.1, $|c(S)| \leqq \|S\|$, which proves the result.

COROLLARY 2.3. *If* $P=\prod_k A_k$, *then* $|c(P)| \leqq \prod_k \|A_k\|$.

Proof. By (iv) $\|P\| \leqq \prod_k \|A_k\|$. So $|c(P)| \leqq \prod_k \|A_k\|$.

In this corollary the matrices A_k need not all be square. Their product P need only be a square matrix.

In Theorem 1 we have mentioned some instances of semimoduli of a matrix. The above corollaries together with Theorem 1 bring out a number of known results. We mention a few by way of illustration. Take $\|A\|$ to be $R(A)$ and $C(A)$. Then Corollary 2.3 gives, when $P=\prod_k A_k$,

$$|c(P)| \leqq \min\left[\prod R(A_k), \prod C(A_k)\right]. \tag{4}$$

Now take $\|A\|$ to be $|A|_0$. Then

$$|c(P)|^2 \leqq \prod\left[c_{\max}(A_k^* A_k)\right]. \tag{5}$$

It is possible to obtain a lower bound for $|c(P)|$ in (5). It can be seen that $|A|_0$ is not a semimodulus of A. For, although $|A|_0$ satisfies (i) and (ii), it may not always satisfy (iii) and (iv). But $|A|_0$ satisfies (iii)′: $|A+B|_0 \leqq \min\left[|A|_0+|B|_0,\ |A|_0+|B|_0\right]$ and (iv)′: $|AB|_0 \geqq |AB|_0 \geqq |A|_0 \cdot |B|_0$. To see this, let x be unit-length ch. vector of B^*B such that $|B|_0 = |Bx|$. Now $|A+B|_0 \leqq |(A+B)x| = |Ax+Bx| \leqq |A|_0+|B|_0$ by (2). Similarly, $|A+B|_0 \leqq |A|_0+|B|_0$. This proves (iii)′. To prove (iv)′: Let y be a unit-length ch. vector of $(AB)^*AB$ such that $|AB|_0 = |ABy|$. Hence $|AB|_0 = |Az| = k|A\bar{z}| \geqq k|A|_0 \geqq |B|_0 \cdot |A|_0$, where $k=|By| = |z| = k|\bar{z}|$, $\bar{z}$ is a unit-length vector, k being zero if $z=0$. So (iv)′ is satisfied. Hence the following theorem is proved.

THEOREM 3. *If* $P=\prod_k A_k$ *is a square matrix then*

$$\prod_k c_{\min}(A_k^* A_k)\left] \leqq |c(P)|^2 \leqq \prod_k \right[c_{\max}(A_k^* A_k)\right].$$

The matrices A_k *need not be all square.*

The above result is a generalization of some results due to Roy [5], Nagy [4], Afriat [1] and others. The following particular cases deserve attention.

COROLLARY 3.1. *When* AB *is a square matrix,*

I $c_{\min}(A^*A)c_{\min}(B^*B) \leqq |c(AB)|^2 \leqq c_{\max}(A^*A)c_{\max}(B^*B)$.

II *When* A *and* B *are normal of the same order,*

$$|c(A)|_{\min}|c(B)|_{\min} \leqq |c(AB)| \leqq |c(A)|_{\max}|c(B)|_{\max}.$$

Proof. Part I follows from Theorem 3. To get the second part we note that when P is normal the ch. values of P^*P are the square of the moduli of the ch. values of P, *i.e.*, $|P|^0 = |c(P)|_{\max}$ and $|P|_0 = |c(P)|_{\min}$.

Inequality (4) may be improved by a result due to Barankin [2]. We shall only consider the case when P is a product of two matrices.

COROLLARY 3.2. *If AB is a square matrix then*

$$|c(AB)|^2 \leqq \min\left[C(A)R(B)\max_i R_i(A)C_i(B),\ C(B)R(A)\max_j R_j(B)C_j(A)\right].$$

Proof. From (1) we have $R_i(AB) \leqq R_i(A)R(B)$. Similarly it can be shown that $C_i(AB) \leqq C(A)C_i(B)$. By Barankin's result

$$|c(AB)|^2 \leqq \max_i R_i(AB)\cdot C_i(AB). \tag{6}$$

Hence

$$|c(AB)|^2 \leqq C(A)R(B)\max_i R_i(A)C_i(B).$$

Also,

$$|c(BA)|^2 \leqq C(B)R(A)\max_j R_j(B)C_j(A).$$

Since the ch. values of AB and BA are the same the result is proved.

LEMMA 1. $\operatorname{Re} c(A) \leqq \|A_1\|$; $\operatorname{Im} c(A) \leqq \|A_2\|$, *where A_1 and A_2 are the hermitian components of A.*

Proof. By a result due to Bromwich [3] we get $\operatorname{Re} c(A) \leqq c_{\max}(A_1)$; $\operatorname{Im} c(A) \leqq c_{\max}(A_2)$. But by Corollary 2.1, $|c_{\max}(A_1)| \leqq \|A_1\|$; $|c_{\max}(A_2)| \leqq \|A_2\|$.

THEOREM 4. *If A and B are complex square matrices of the same order then*

$$\operatorname{Re} c(A \pm B) \leqq \|A_1\| + \|B_1\|, \qquad \operatorname{Im} c(A \pm B) \leqq \|A_2\| + \|B_2\|.$$

Proof. $A \pm B = A_1 \pm B_1 + i(A_2 \pm B_2) = S = S_1 + iS_2$, say. Thus S_1 and S_2 are the hermitian components of S. So

$$\operatorname{Re} c(S) = \operatorname{Re} c(A \pm B) \leqq \|S_1\| = \|A_1 \pm B_1\| \leqq \|A_1\| + \|B_1\|,$$
$$\operatorname{Im} c(S) = \operatorname{Im} c(A \pm B) \leqq \|S_2\| = \|A_2 \pm B_2\| \leqq \|A_2\| + \|B_2\|.$$

COROLLARY 4.1.

(a) $\operatorname{Re} c(A + B) \leqq \frac{1}{2}[R(A + A^*) + R(B + B^*)]$,

(b) $\operatorname{Im} c(A + B) \leqq \frac{1}{2}[R(A - A^*) + R(B - B^*)]$,

(c) $\operatorname{Re} c(A + B) \leqq \frac{1}{2}[|c(A + A^*)|_{\max} + |c(B + B^*)|_{\max}]$,

(d) $\operatorname{Im} c(A + B) \leqq \frac{1}{2}[|c(A - A^*)|_{\max} + |c(B - B^*)|_{\max}]$.

Proof. (a) and (b) follow since $R(A)$ is a semimodulus. A similar result will also hold for $C(A)$. Results (c) and (d) follow from the fact that $|A|^0$ is a semimodulus and, when P is normal, $|P|^0 = |c(P)|_{\max}$.

THEOREM 5. *If* $AB = BA$ *then*

$$\operatorname{Re} c(AB) \leqq \|A_1\| \cdot \|B_1\| + \|A_2\| \cdot \|B_2\|, \qquad \operatorname{Im} c(AB) \leqq \|A_1\| \cdot \|B_2\| + \|A_2\| \cdot \|B_1\|.$$

Proof. $AB = A_1B_1 - A_2B_2 + i(A_1B_2 + A_2B_1)$ while we have $BA = B_1A_1 - B_2A_2 + i(B_2A_1 + B_1A_2)$. Since $AB = BA$ we have $A_1B_1 - A_2B_2 = B_1A_1 - B_2A_2 = (A_1B_1 - A_2B_2)^*$. So $P_1 = A_1B_1 - A_2B_2$ is hermitian. Similarly, $P_2 = A_1B_2 + A_2B_1$ is hermitian. Hence if $P = AB$ then P_1 and P_2 are the hermitian components of P and the result follows by Lemma 1.

COROLLARY 5.1. *When* A *and* B *commute*

$$\operatorname{Re} c(AB) + \operatorname{Im} c(AB) \leqq [\|A_1\| + \|A_2\|] \cdot [\|B_1\| + \|B_2\|].$$

COROLLARY 5.2. *When* A *and* B *commute*

(a) $\operatorname{Re} c(AB) \leqq \frac{1}{4}[R(A + A^*)R(B + B^*) + R(A - A^*)R(B - B^*)]$,

(b) $\operatorname{Im} c(AB) \leqq \frac{1}{4}[R(A + A^*)R(B - B^*) + R(A - A^*)R(B + B^*)]$,

(c) $\operatorname{Re} c(AB) \leqq \frac{1}{4}[|c(A + A^*)|_{\max} |c(B + B^*)|_{\max} + |c(A - A^*)|_{\max} |c(B - B^*)|_{\max}]$,

(d) $\operatorname{Im} c(AB) \leqq \frac{1}{4}[|c(A + A^*)|_{\max} |c(B - B^*)|_{\max} + |c(A - A^*)|_{\max} |c(B + B^*)|_{\max}]$.

Proof. The proof is similar to that of Corollary 4.1.

My thanks are due to Professor S. M. Shah for his suggestions.

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MATHEMATICAL NOTES

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L'HOSPITAL'S RULE FOR COMPLEX-VALUED FUNCTIONS

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L'Hospital's rule for real functions may be stated in the form:

Let $f(x)$ and $g(x)$, and their derivatives $f'(x)$, $g'(x)$, be continuous on an open interval $(0, a)$, and let

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0.$$

*If the ratio f'/g' is defined on $(0, a)$ (in the sense that g' does not vanish) and has a finite limit at $x=0$, then the ratio of functions, f/g, is defined and has the same limit.**

A simple example shows that this rule is not generally valid when f and g are complex-valued functions of a real variable. Taking $f=x$, $g=xe^{-i/x}$, we have

$$f'/g' = \frac{xe^{i/x}}{x+i},$$

which vanishes as $x\to 0$, while $f/g=e^{i/x}$ has no limit. The more pathological example $f=x$, $g=x(e^{-i/x}-1)$ shows that f/g need not even be defined.

However, by placing additional restrictions on the functions, we can obtain generalizations of the rule which cover many cases. For example, the rule is valid provided

I. *The ratio $|g'|/|g|'$ is defined (in the sense that the derivative $|g|'$ of $|g|$ exists and does not vanish) and bounded on $(0, a)$.*

This is the simplest of a class of conditions obtained by introducing the difference, $\delta(x)$, between f'/g' and its limit

$$L = \lim_{x\to 0} f'(x)/g'(x).$$

Multiplying the equation $\delta(x)=f'(x)/g'(x)-L$ by $g'(x)$ and integrating, we have

$$f(x) - Lg(x) = \int_0^x \delta(y)g'(y)dy,$$

in which the integral may be improper. On dividing by $g(x)$, it is clear that f/g has the limit L provided

* This rule is equally valid if f'/g' has an infinite limit at $x=0$, since the conditions on f and g are the same. This is not true of the conditions derived below for complex-valued functions. Here the roles of f and g must be interchanged explicitly to cover cases of infinite limits.

II. *The ratio*

$$R(x) \equiv \left[\int_0^x \delta(y)g'(y)dy\right] \Big/ g(x)$$

is defined on $(0, a)$ *and vanishes as* $x \to 0$. We will show that each of the two following conditions, as well as I, implies Condition II and is therefore sufficient for validity:

III. *The ratio* $r(x) \equiv |\delta(x)||g'(x)| / |g(x)|'$ *is defined on* $(0, a)$ *and vanishes as* $x \to 0$, *where* $|g|'$ *is the derivative of* $|g|$.

IV. $|g(x)|$ *is monotone and the real and imaginary parts of* $\delta(x)$ *are of bounded variation on* $(0, a)$.

The first step in the proof is to show that g cannot vanish on $(0, a)$. For Conditions I and III this follows immediately from Rolle's theorem and the fact that $|g|$ is continuous and vanishes at $x=0$, while $|g|'$ is defined and does not vanish. For Condition IV it follows from the monotonicity of $|g|$: If $g(\xi)=0$, $0<\xi<a$, then g vanishes identically on $(0, \xi)$, which contradicts the hypothesis that $g' \not\equiv 0$.

To show that III implies II, we note that $|g|'$ is given by $|g|' = (g\bar{g}' + g'\bar{g}) / 2|g|$, where the complex conjugate $\bar{g}$ enjoys the same continuity properties as g. Hence $|g|'$ is continuous; and if we define $r(0)=0$, it follows that $r(x)$ is continuous on the closed interval $[0, a/2]$. Let ρ be an upper bound for r on that interval. Then

$$\int_\epsilon^x |\delta(y)|\,|g'(y)|\,dy \leqq \rho \int_\epsilon^x |g(y)|'dy < \rho\,|g(x)|,$$

and on taking the limit $\epsilon \to 0$ we see that

$$\int_0^x |\delta(y)|\,|g'(y)|\,dy$$

is defined, continuous, and has the continuous derivative $|\delta||g'|$ on $(0, a/2)$. Hence the ratio

$$R_1(x) = \left[\int_0^x |\delta(y)|\,|g'(y)|\,dy\right] \Big/ |g(x)|$$

satisfies the conditions of l'Hospital's rule for real functions. Applying the rule to R_1, we find that Condition II is satisfied, since $|R| \leqq |R_1|$.

Condition I clearly implies III, since $\delta(x)$ vanishes as $x \to 0$. Hence I implies II.

To show that IV implies II, we integrate the numerator of $R(x)$ by parts and use the monotonicity of $|g|$ to obtain the estimate

$$|R(x)| \leqq |\delta(x)| + \int_0^x |d\delta(y)|.$$

Hence R vanishes with x and II is satisfied.

Similar arguments show that Conditions I, III, and IV are also valid for the Stolz extension of l'Hospital's rule, in which the conditions

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$$

are replaced by

$$\lim_{x\to 0} |f(x)| = \lim_{x\to 0} |g(x)| = \infty.$$

ON A DETERMINANTAL INEQUALITY

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Recently L. K. Hua [2] proved the following interesting inequality: *Let A and B be n-square complex matrices and assume*

$$I - A^*A \quad \text{and} \quad I - B^*B \tag{1}$$

are both positive semidefinite. Then

$$|d(I - A^*B)|^2 \geqq d(I - A^*A)d(I - B^*B) \tag{2}$$

where d is the determinant and A^* is the conjugate transpose of A.

We prove here an extension of the inequality (2). Let $\lambda_j, \alpha_j, \beta_j$ be respectively the eigenvalues of $I-A^*B$, A^*A and B^*B so indexed that

$$|\lambda_j| \geqq |\lambda_{j+1}|, \qquad \alpha_j \geqq \alpha_{j+1}, \qquad \beta_j \geqq \beta_{j+1} \qquad \text{for } j = 1, \cdots, n-1.$$

THEOREM. *If $I-A^*A$ and $I-B^*B$ are both positive semidefinite then for each k satisfying $1 \leqq k \leqq n$,*

$$\prod_{j=1}^{k} |\lambda_{n-j+1}|^2 \geqq \prod_{j=1}^{k} (1 - \alpha_j)(1 - \beta_j). \tag{3}$$

We first establish an inequality.

LEMMA. *If u and v are complex n vectors and*

$$\|u + v\| \leqq 2 \tag{4}$$

then

$$|1 - (u, v)|^2 \geqq (1 - \|u\|^2)(1 - \|v\|^2). \tag{5}$$

* This work was completed under a National Research Council-National Bureau of Standards Postdoctoral Research Associateship. The author also wishes to acknowledge the helpful suggestions of Dr. Olga Taussky-Todd.

Proof. Let $u \wedge v$ denote the Grassmann exterior product of u and v. Then it is known ([4], p. 63) that

$$\|u \wedge v\|^2 = \|u\|^2\|v\|^2 - |(u, v)|^2. \tag{6}$$

Now we assert that in general

$$\|u \wedge v\|^2 \leqq \tfrac{1}{4}\|u - v\|^2\|u + v\|^2. \tag{7}$$

For set $x=u+v$, $y=u-v$ and note that

$$u \wedge v = \left(\frac{x+y}{2}\right) \wedge \left(\frac{x-y}{2}\right) = (1/4)(x \wedge x - x \wedge y + y \wedge x - y \wedge y)$$
$$= (1/2)(y \wedge x).$$

Hence by (6)

$$\|u \wedge v\|^2 = (1/4)(y \wedge x, y \wedge x) = (1/4)(\|y\|^2\|x\|^2 - |(x, y)|^2)$$
$$\leqq (1/4)\|x\|^2\|y\|^2 = (1/4)\|u - v\|^2\|u + v\|^2.$$

It follows from (4) and (7) that

$$\|u\|^2\|v\|^2 - |(u, v)|^2 \leqq \|u - v\|^2 = \|u\|^2 + \|v\|^2 - [(u, v) + (v, u)].$$

Hence

$$1 - [(u, v) + (v, u)] + |(u, v)|^2 \geqq 1 - (\|u\|^2 + \|v\|^2) + \|u\|^2\|v\|^2$$

and this is precisely the inequality (5). Proceeding to the proof of (3) we next note that if x is a vector with $\|x\| = 1$ then by (1)

$$\|Ax + Bx\| \leqq \|Ax\| + \|Bx\|$$
$$= (A^*Ax, x)^{1/2} + (B^*Bx, x)^{1/2} \leqq 2.$$

Hence by (5) with $u=Ax$ and $v=Bx$

$$|([I - A^*B]x, x|^2 = |1 - (Bx, Ax)|^2 \geqq (1 - \|Ax\|^2)(1 - \|Bx\|^2) \tag{8}$$
$$= (1 - (A^*Ax, x))(1 - (B^*Bx, x)) = ([I - A^*A]x, x)([I - B^*B]x, x).$$

By a theorem of Schur [3] there is an orthonormal set of vectors $x_1, \cdots, x_k$ for which $([I-A^*B]x_j, x_j) = \lambda_{n-j+1}$ for $j=1, \cdots, k$. Then from (8) we have

$$\prod_{j=1}^{k} |\lambda_{n-j+1}|^2 = \prod_{j=1}^{k} |(I - A^*B]x_j, x_j)|^2 \tag{9}$$
$$\geqq \prod_{j=1}^{k} ([I - A^*A]x_j, x_j)([I - B^*B]x_j, x_j).$$

A theorem of Ky Fan [1] states that *if H is positive semidefinite Hermitian then*

$$(10) \qquad \prod_{j=1}^{k} (Hx_j, x_j) \geqq \prod_{j=1}^{k} h_{n-j+1}$$

where $h_1 \geqq \cdots \geqq h_n$ *are the eigenvalues of* H *and* $x_1, \cdots, x_k$ *is a set of* k *orthonormal vectors*. We see then that (3) follows immediately from (9) and (1), and the proof is complete.

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APPROXIMATING A CIRCULAR SEGMENT BY USE OF DIOPHANTINE EQUATIONS

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The purpose of this note is to obtain a simple rational approximation for the ratio $G = S/T$ of the area S of a segment of a circle to the area T of the inscribed isosceles triangle, as a function of the ratio $h = H/R$ of the height H of the segment to the radius R of the circle. This approximation is uniformly more accurate for $0 < h \leqq 1$ than the approximation $22/7$ for π. Yet it involves few terms or factors and only relatively small integral coefficients.

Let $2\theta (\leqq \pi)$ be the radian measure of the central angle that subtends the bounding arc and bounding chord of the given segment. Then the arc length and chord length are $2R\theta$ and $2R \sin \theta$, respectively, and the arc-chord ratio is the function $F = \theta/\sin \theta$. The areas S of the segment and T of the inscribed isosceles triangle, and their common height Rh are

$$(1) \qquad S = R^2(\theta - \sin \theta \cos \theta), \qquad T = R^2 \sin \theta(1 - \cos \theta), \qquad Rh = R(1 - \cos \theta).$$

The segment-triangle ratio G and the arc-chord ratio F are the following functions of h:

$$(2) \qquad S/T = G(h) = 1 + (F - 1)/h,$$

$$(3) \qquad F(h) = \theta/\sin \theta = 2 \arcsin (h/2)^{1/2}/(2h - h^2)^{1/2}.$$

To obtain a series for F in powers of h we may differentiate $(\sin \theta)F - \theta$ with respect to θ and replace $\cos \theta$ by $1 - h$. We find

$$(4) \quad (\sin^2 \theta)(dF/dh) + (\cos \theta)F - 1 = 0, \qquad (2h - h^2)F' + (1 - h)F - 1 = 0,$$

$$(5) \qquad 2hF' + F - 1 = h^2F' + hF.$$

We substitute in (5) a power series of the form

$$F = a_0 + a_1h + a_2h^2 + a_3h^3 + \cdots, \tag{6}$$

and obtain the recurrence relations

$$(2n + 1)a_n = na_{n-1}, \qquad a_0 = 1. \tag{7}$$

Hence the functions $F(h)$ and $G(h)$ have the power series expansions

$$F(h) = \theta/\sin\theta = 1 + h/3 + 1\cdot 2h^2/3\cdot 5 + 1\cdot 2\cdot 3h^3/3\cdot 5\cdot 7 + \cdots, \tag{8}$$

$$G(h) = 1 + (F - 1)/h = 1 + 1/3 + 1\cdot 2h/3\cdot 5 + 1\cdot 2\cdot 3h^2/3\cdot 5\cdot 7 + \cdots, \tag{9}$$

which for $h=1$ give the same series for $\pi/2$:

$$\pi/2 = F(1) = 1 + 1/3 + 1\cdot 2/3\cdot 5 + 1\cdot 2\cdot 3/3\cdot 5\cdot 7 + \cdots. \tag{10}$$

Since ten terms of the series (10) must be used to reduce the remainder to about 0.0005, the series (9) does not of itself yield a polynomial approximation of low degree with a uniform error less than 0.0005.

We require, if possible, a simple rational function $f(h)$ such that the function $G(h) = S/T$ in (2) is approximated uniformly for $0 < h \leqq 1$, with a maximum error of 0.0005, by

$$g(h) = 1 + (f(h) - 1)/h. \tag{11}$$

Clearly we must require that $f(0) = 1$, in order for (11) to be finite at $h=0$. Although the approximation

$$f_1(h) = 1/(1 - h/3), \quad \text{or} \quad \theta \sim 3\sin\theta/(2 + \cos\theta), \tag{12}$$

gives the angle θ to within one minute for angles less than 37°, it is always too small and differs from $F(h)$ by 0.071 for $h=1$. It can be shown that no approximation of the form $1/(1-bh)$ can differ from $F(h)$ by less than 0.021 for all h between 0 and 1. The more complicated approximation

$$f_2(h) = 1 + h/(3 - h - h/4), \text{ or } \theta \sim \sin\theta(11 + \cos\theta)/(7 + 5\cos\theta), \tag{13}$$

is considerably closer than (12), since it gives 11/7 for $\pi/2$ at $h=1$, and has a maximum error of about 0.003 near $h = 2/3$.

But to attain the desired accuracy, we try a three parameter approximation of the form

$$f(h) = a/(1 - bh) + (1 - a)/(1 - ch), \qquad a, b, c \text{ rational}, \tag{14}$$

and we first demand that its power series about $h=0$ agree with (8) to terms in h^2. Thus we are led to the pair of Diophantine equations

$$ab + (1 - a)c = 1/3, \tag{15}$$

$$ab^2 + (1 - a)c^2 = 2/15. \tag{16}$$

Squaring (15) and subtracting from (16) we derive the relations:

$$a(1 - a)(b - c)^2 = 2/15 - 1/9 = 1/45, \tag{17}$$

$$[5(1-a)/a][3a(b-c)]^2 = 1. \tag{18}$$

Denoting $3a(b-c)$ by $1/u$, and solving (18) and (15) for a, b, c we obtain the parametric solution

$$a = 5/(5+u^2), \qquad 3b = 1 + u/5, \qquad 3c = 1 - 1/u. \tag{19}$$

Next we demand that u be chosen in (19) so that $f(1)$ in (14) is a close approximation to $\pi/2$. Setting

$$f(1) = \frac{\pi}{2} = \frac{5/(5+u^2)}{2/3 - u/15} + \frac{u^2/(5+u^2)}{2/3 + 1/3u}, \tag{20}$$

we find on simplification that

$$\frac{\pi}{3} - 1 = \frac{1}{(10-u)(2+1/u)}, \tag{21}$$

$$(10-u)(2+1/u) = 3/(\pi - 3) = 21.1876, \cdots, \quad u = 1.755 \cdots \tag{22}$$

Finally we pick the simple rational value $u = 7/4$ near $1.755 \cdots$ and obtain from (19) and (20) the rational solution

$$a = 80/129, \quad b = 9/20, \quad c = 1/7, \quad f(1) = 311/198 = 1.5707 \cdots. \tag{23}$$

Thus we obtain the desired approximation formulas

$$\frac{\theta}{\sin\theta} = \frac{1}{129}\left(\frac{80}{1 - 9h/20} + \frac{49}{1 - h/7}\right) = \frac{420 - 109h}{(20 - 9h)(21 - 3h)} = \frac{311 + 109\cos\theta}{(11 + 9\cos\theta)(18 + 3\cos\theta)}, \tag{24}$$

$$\frac{\text{Segment}}{\text{Triangle}} = 1 + \frac{1}{129}\left(\frac{36}{1 - 9h/20} + \frac{7}{1 - h/7}\right) = 1 + \frac{140 - 27h}{(20 - 9h)(18 - 3h)}, \tag{25}$$

$$0 < h = 1 - \cos\theta \leqq 1.$$

For small h these formulas give values too large by about $h^3/2100$ and $h^2/2100$ respectively. The maximum errors of 0.00017 and 0.00023 respectively are found near $\theta = 75°$. Both errors vanish near $\theta = 88°$ and become about -0.00009 at $\theta = 90°$, $h = 1$.

In conclusion we state without proof that $G(h)$ has the continued fraction expansion

$$G(h) - 1 = (F(h) - 1)/h = \cfrac{1}{3 - h - \cfrac{h}{5 - \cfrac{6h}{7 - R}}}, \quad \text{where} \quad R = \cfrac{6h}{9 - \cfrac{15h}{11 - \cfrac{15h}{13 - \cdots}}}, \tag{26}$$

and that formulas (24) and (25) result from (26) when R is replaced by 7/9.

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A FIXED POINT THEOREM

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By the term *continuum* we mean a compact connected metric space. A continuum M is *chainable* if, for each $\epsilon>0$, there is a finite collection of open sets, $U_1, \cdots, U_{n(\epsilon)}$, with the following properties.

(i) *diam* $U_i<\epsilon$ *for each* i,
(ii) $M\subset\cup\{U_i: i=1, \cdots, n(\epsilon)\}$,
(iii) $\overline{U}_i$ *meets* $\overline{U}_j$ *if and only if* $|i-j|\leqq 1$,
(iv) *for each* i, $U_i-\cup\{\overline{U}_j: j\neq i\}$ *is not empty.*

The simplest nontrivial example of a chainable continuum is the closed unit interval of real numbers. However, Bing [2] has exhibited a very pathological chainable continuum, the so-called pseudo-arc.

A space M has the f.p.p. (=fixed point property) if, for each continuous function $f: M\to M$ there exists some $x\in M$ such that $x=f(x)$. Hamilton [1] has proved that the chainable continua have the f.p.p.

Let $F: M\to M$ be a multivalued function, *i.e.*, $F(x)$ is a subset of M for each $x\in M$. Strother [3] has defined such a mapping to be continuous if each $F(x)$ is a closed set and, whenever $x_n\to x_0$ in M, the following two conditions are satisfied:

(a) *if* $y_n\in F(x_n)$ *then* $\{y_n\}$ *has a limit point in* $F(x_0)$,
(b) *if* $y_0\in F(x_0)$ *then there exists* $y_n\in F(x_n)$ *such that* $y_n\to y_0$.

The space M is said to have the F.p.p. if each continuous multivalued function $F: M\to M$ has a fixed point, *i.e.*, $x\in F(x)$ for some $x\in M$. It is the purpose of this note to generalize Hamilton's theorem to the multivalued case.

THEOREM. *Each chainable continuum has the* F.p.p.

Proof. Let M be a chainable continuum and let $F: M\to M$ be a continuous multivalued function. In view of (a) it is clearly sufficient to prove that, for $\epsilon>0$, there exists $\bar{x}\in M$ such that $d(\bar{x}, y)<\epsilon$ for some $y\in F(\bar{x})$. Accordingly, we select a collection of open sets, $U_1, \cdots, U_{n(\epsilon)}$, satisfying conditions (i)–(iv)

above. Let A be the set of all $x \in M$ such that for some i, $x \in \overline{U}_i$ and $F(x) \cap U_j$ is empty for $j<i$; we define B to be the set of all $x \in M$ such that $x \in \overline{U}_i$ implies that $F(x)$ meets $\overline{U}_j$ for some $j<i$. It is obvious that $A \cup B = M$ and that $\overline{U}_1 \subset A$. If $A = M$, then the $\bar{x}$ we seek may be any point in $U_{n(\epsilon)}$. Otherwise, B is non-empty and from continuity we infer that A and B are closed sets. Since M is connected, there is a point, $\bar{x}$, common to A and B. It follows that, for some i, $\bar{x} \in \overline{U}_i \cap \overline{U}_{i+1}$ and there exists $y \in F(\bar{x}) \cap U_i$. Since $d(\bar{x}, y) < \epsilon$, the proof is complete.

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CURVE-FITTING MATRICES

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1. Introduction. Suppose we are given N function values $y_1, \cdots, y_N$ at N equally spaced points $x_1, \cdots, x_N$, where $x_{n+1} - x_n = h$. Let it be desired to find the coefficients of the polynomial

$$p(x) = a_0 + a_1 x + \cdots + a_{N-1} x^{N-1} \tag{1}$$

such that

$$p(x_i) = y_i \qquad (i = 1, \cdots, N). \tag{2}$$

The conditions (2) are a set of N equations in the unknowns a_j of the form

$$V_N A = \pi, \tag{3}$$

where $A = \{a_0, a_1, \cdots, a_{N-1}\}$, $\pi = \{y_1, \cdots, y_N\}$ and V_N is the Vandermonde matrix of order N

$$(V_N)_{ij} = x_i^{j-1} \qquad (i, j = 1, \cdots, N). \tag{4}$$

A change of origin and scale

$$s = (x - x_1)/(x_N - x_1) \tag{5}$$

reduces (4) to the form

$$(V_N)_{ij} = s_i^{j-1} = \left[\frac{i-1}{N-1}\right]^{j-1} \qquad (i, j = 1, \cdots, N). \tag{6}$$

The unknown coefficients A may then be written

$$A = V_N^{-1} \pi \tag{7}$$

with V_N given by (6).

The matrices $\overline{V}_N^{-1}$ may be calculated once and for all, so that curve fitting becomes simply a matrix by vector multiplication. We tabulate below the V_N^{-1} for $N=2, \cdots, 6$.

2. The Matrices $\overline{V}_N^{-1}$.

$$V_2^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \qquad V_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 4 & -1 \\ 2 & -4 & 2 \end{pmatrix},$$

$$V_4^{-1} = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 \\ -11 & 18 & -9 & 2 \\ 18 & -45 & 36 & -9 \\ -9 & 27 & -27 & 9 \end{pmatrix}, \quad V_5^{-1} = \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ -25 & 48 & -36 & 16 & -3 \\ 70 & -208 & 228 & -112 & 22 \\ -80 & 288 & -384 & 224 & -48 \\ 32 & -128 & 192 & -128 & 32 \end{pmatrix},$$

$$V_6^{-1} = \frac{1}{24}\begin{pmatrix} 24 & 0 & 0 & 0 & 0 & 0 \\ -274 & 600 & -600 & 400 & -150 & 24 \\ 1125 & -3850 & 5350 & -3900 & 1525 & -250 \\ -2125 & 8875 & -14750 & 12250 & -5125 & 875 \\ 1875 & -8750 & 16250 & -15000 & 6875 & -1250 \\ -625 & 3125 & -6250 & 6250 & -3125 & 625 \end{pmatrix}.$$

3. An example. We consider the problem of fitting a curve of fifth degree to a table of values of $k=c_p/c_v$ versus temperature in air at low pressure. The table of values is [1]:

T (F Abs.)	k
600°	1.399
900°	1.387
1200°	1.368
1500°	1.350
1800°	1.336
2100°	1.325

We find easily that

$$V_6^{-1}\begin{pmatrix}1.399\\1.387\\1.368\\1.350\\1.336\\1.325\end{pmatrix}=\begin{pmatrix}1.399000\\-.021916\\-.255208\\.359375\\-.182292\\.026042\end{pmatrix}$$

so that

$$k=f(u)=1.399-.021916u-.255208u^2+.359375u^3-.182292u^4+.026042u^5$$

where $u=(T-600°)/1500°$ is the desired fit.

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CLASSROOM NOTES

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A UNIFYING TECHNIQUE FOR THE SOLUTION OF THE QUADRATIC, CUBIC, AND QUARTIC

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In this paper, starting with Gauss's fundamental theorem of algebra and its immediate extension that every algebraic equation of degree n has n roots, the solutions of the quadratic, cubic, and quartic are derived. The only other results used are the relations between the roots and coefficients of the equations. The solution in the cubic case is identical with Cardan's and, in the quartic case, the result is the same as that obtained by the method of Descartes. In each case, the motivation for the solution is clearer than in conventional methods. The quadratic case, which is very simple, is included to show the unifying technique and, in order to obtain the familiar formula, the leading coefficient will be taken to be a, but in the case of the cubic and the quartic, the equations will be in reduced form with leading coefficients unity.

Case I ($n=2$). The general quadratic equation is $ax^2+bx+c=0$, $a\neq 0$. Let r_1 and r_2 be its two roots so that

$$r_1 + r_2 = -b/a, \qquad r_1 r_2 = c/a. \tag{1}$$

Then

$$(r_1 - r_2)^2 = \frac{b^2}{a^2} - \frac{4c}{a}. \tag{2}$$

Extracting the square root of (2) and solving with the first equation of (1) results in

$$r_1 = (-b \pm \sqrt{b^2 - 4ac})/(2a), \qquad r_2 = (-b \mp \sqrt{b^2 - 4ac})/(2a).$$

These give permuted values for r_1 and r_2, and the two roots are given by $x = (-b \pm \sqrt{b^2 - 4ac})/(2a)$.

Case II ($n = 3$). It is sufficient to consider the reduced cubic

$$x^3 + ax + b = 0. \tag{3}$$

Let r_1, r_2, and r_3 be its three roots so that

$$r_1 + r_2 + r_3 = 0, \tag{4}$$

$$r_1 r_2 + r_1 r_3 + r_2 r_3 = a, \tag{5}$$

$$r_1 r_2 r_3 = -b. \tag{6}$$

Substituting r_1, r_2, and r_3 successively in (3), adding the resulting equations, and using (4) and (6) yields $r_1^3 + r_2^3 + r_3^3 - 3r_1 r_2 r_3 = 0$, which factors into

$$(r_1 + r_2 + r_3)(r_1 + \omega r_2 + \omega^2 r_3)(r_1 + \omega^2 r_2 + \omega r_3) = 0, \tag{7}$$

where ω and ω^2 are the imaginary cube roots of unity. From (7) we have the three possibilities $r_1 = -(r_2 + r_3)$, $r_1 = -(\omega r_2 + \omega^2 r_3)$, $r_1 = -(\omega^2 r_2 + \omega r_3)$. These results suggest that the three roots of (3) can be found by setting

$$r_1 = -(s + t), \qquad r_2 = -(\omega s + \omega^2 t), \qquad r_3 = -(\omega^2 s + \omega t) \tag{8}$$

and determining s and t so that (8) satisfies (4), (5), and (6).

Now (4) is satisfied for all s and t since $\omega^2 + \omega + 1 = 0$; therefore, (5) and (6) become, respectively,

$$st = -a/3, \tag{9}$$

$$s^3 + t^3 = b. \tag{10}$$

Raising (9) to the third power and solving simultaneously with (10), using the method of Case I, yields

$$s = \left\{\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{1/3}, \qquad t = \left\{\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{1/3}.$$

The first equation of (8) then becomes

$$r_1 = \left\{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{1/3} + \left\{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right\}^{1/3}.$$

which is Cardan's result. The other two roots are given by the second and third equations of (8).

Case III ($n=4$). Here it is sufficient to consider $x^4+ax^2+bx+c=0$ whose roots will be denoted by r_1, r_2, r_3, and r_4. Then

$$\begin{aligned} r_1 + r_2 + r_3 + r_4 &= 0, \\ r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4 &= a, \\ r_1r_2r_3 + r_1r_2r_4 + r_2r_3r_4 + r_1r_3r_4 &= -b, \\ r_1r_2r_3r_4 &= c. \end{aligned} \tag{11}$$

Making the substitutions

$$u = r_1 + r_2, \quad v = r_3 + r_4, \quad s = r_1r_2, \quad t = r_3r_4 \tag{12}$$

in equations (11) yields

$$u + v = 0, \quad s + t + uv = a, \quad sv + tu = -b, \quad st = c.$$

These reduce to

$$s + t = a + u^2, \tag{13}$$

$$u(s - t) = b, \tag{14}$$

$$st = c. \tag{15}$$

Solving (13) and (14) for s, t in terms of u, a, and b results in

$$s = \frac{1}{2}\left\{a + u^2 + \frac{b}{u}\right\}, \quad t = \frac{1}{2}\left\{a + u^2 - \frac{b}{u}\right\}. \tag{16}$$

Substituting s, t from (16) in (15) and simplifying leads to

$$u^6 + 2au^4 + (a^2 - 4c)u^2 - b^2 = 0. \tag{17}$$

This is a cubic equation in u^2 and can be solved by the method of Case II; from this u is found and hence v. Using the values of u, s and v, t in pairs according to equation (12) and the technique of Case I determines r_1, r_2, r_3, and r_4.

Equation (17) is identical with that obtained by the method of Descartes for the solution of the quartic.

NOTE ON A REMARK OF WALD

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In Part II, applications of the general theory to special cases, of *Sequential Analysis*, Wald considers the following simple problem in setting up the sequential probability ratio test for the acceptance inspection of a lot where each unit is classified into one of two categories. Let p be the unknown proportion of defectives in the lot. Usually it is possible to specify p_0 and p_1 such that we prefer acceptance if $p \leqq p_0$, prefer rejection if $p_1 \leqq p$, and will be indifferent if $p_0 < p < p_1$, where p_0, p, and p_1 are all between 0 and 1. Given α and β, the type I and type II errors respectively, the approximate equations of lines that divide the sample space into three mutually exclusive sets for the decisions described above are given by:

$$(1) \qquad a_m = \frac{\log \dfrac{\beta}{1-\alpha}}{\log \dfrac{p_1}{p_0} - \log \dfrac{1-p_1}{1-p_0}} + m \frac{\log \dfrac{1-p_0}{1-p_1}}{\log \dfrac{p_1}{p_0} - \log \dfrac{1-p_1}{1-p_0}},$$

$$(2) \qquad r_m = \frac{\log \dfrac{1-\beta}{\alpha}}{\log \dfrac{p_1}{p_0} - \log \dfrac{1-p_1}{1-p_0}} + m \frac{\log \dfrac{1-p_0}{1-p_1}}{\log \dfrac{p_1}{p_0} - \log \dfrac{1-p_1}{1-p_0}},$$

i.e.,

$$(3) \qquad a_m = h_1 + ms, \qquad r_m = h_2 + ms,$$

where a_m and r_m are respectively the acceptance and rejection numbers, and h_1, h_2, s stand for the corresponding quantities in (1) and (2) above. Clearly, s the slope being the same, the lines for (a_m, m) and (r_m, m) are parallel. The quantity s is important in the test procedure, since the range of its variability with respect to p_0, p_1 is usually of interest; thus, we consider it here.

On page 100 in the footnote, Wald remarks that it can be shown that s lies between p_0 and p_1, the quantities that appear in (1), (2) and (3) above. The proof of this statement is not given there. A. J. Duncan, in his book, has given a proof in a mathematical appendix (p. 580), using the first mean value theorem for integrals.

Since in part II Wald has limited himself to as simple a mathematics as possible, one feels that it must be possible to give a simpler proof of this remark than the one given in Duncan's book. It is the purpose of this small note to supply such a proof.

We now prove that:

If $$s = \frac{\log \dfrac{1-p_0}{1-p_1}}{\log \dfrac{p_1}{p_0} - \log \dfrac{1-p_1}{1-p_0}}$$ *where* $0 < p_0 < p_1 < 1$, *then* $p_0 < s < p_1$.

Proof. Given that $p_0 < p_1$, so that $p_0(1-p_1) < p_1(1-p_0)$ or

$$\left(\frac{1-p_1}{1-p_0}\right)^{p_1-p_0} < \left(\frac{p_1}{p_0}\right)^{p_1-p_0} \tag{4}$$

since $p_1 - p_0 > 0$, and the inequality is unaltered; that is,

$$\left(\frac{1-p_1}{1-p_0}\right)^{1-p_0} \times \left(\frac{1-p_0}{1-p_1}\right)^{1-p_1} < \left(\frac{p_1}{p_0}\right)^{p_1}\left(\frac{p_0}{p_1}\right)^{p_0}. \tag{5}$$

Now this inequality can be written as a product of the following two inequalities. Note that no other pair of inequalities whose product is (5) may be formed. Thus:

$$\left(\frac{1-p_1}{1-p_0}\right)^{1-p_0} < \left(\frac{p_0}{p_1}\right)^{p_0}, \tag{6}$$

$$\left(\frac{1-p_0}{1-p_1}\right)^{1-p_1} < \left(\frac{p_1}{p_0}\right)^{p_1}. \tag{7}$$

The motivation in writing this is to separate (5) into p_0 powers and p_1 powers, since we want anyway the ratios p_1/p_0 and $(1-p_0)/(1-p_1)$ in the final result.

Now (6) may be simplified as

$$\left(\frac{1-p_1}{1-p_0}\right) < \left[\frac{p_0(1-p_1)}{p_1(1-p_0)}\right]^{p_0}.$$

Taking logs and then multiplying by minus one, we get

$$\log\left(\frac{1-p_0}{1-p_1}\right) > p_0\left[\log\frac{p_1}{p_0} - \log\left(\frac{1-p_1}{1-p_0}\right)\right]$$

which gives

$$p_0 < \frac{\log\left(\dfrac{1-p_0}{1-p_1}\right)}{\log\dfrac{p_1}{p_0} - \log\left(\dfrac{1-p_1}{1-p_0}\right)} = s.$$

A similar simplification of (7) gives that $p_1 > s$, and combining these two inequalities completes the proof of the remark of Wald.

A NOTE ON BERTRAND'S PROBLEM*

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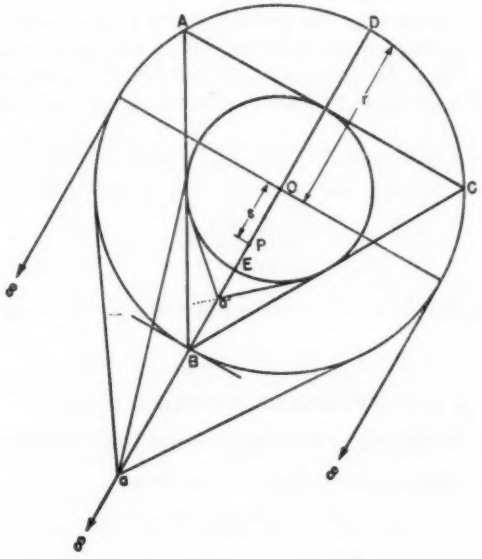
Bertrand's problem has long been known as an example of the fact that a seemingly unambiguous question may turn out upon analysis to have multiple answers according to the particular probability model applied to the problem.

The paradox arises in simple form when we ask the probability of obtaining at random a chord of a circle greater in length than the length of the side of the inscribed equilateral triangle. Two common answers to this problem are evident from the figure. It is clear that if we choose all the chords which pass through a



point on the circumference, say B, 1/3 of them will lie in the angle ABC, and the probability is thus intuitively 1/3. On the other hand, the problem may be approached by considering all chords having their midpoints upon the radius OD and running perpendicular to this line. Here it is intuitively evident that half of the chords will be greater in length than AC, and the probability is thus 1/2.

Of course, both of these answers are correct. It is simply that our random mechanisms differ. The first approach considers the angle at B between the tangent and the chord uniformly distributed between 0 and 180°. The second considers the distance of the midpoint of the chord from the center to be uni-

* Preparation of the manuscript was supported in part by funds from the Research Committee of the University of Alabama.

formly distributed from 0 to r. Other random mechanisms are possible which will yield still other results, pointing up even more sharply the ambiguity of the first statement of the problem. It is interesting, however, to observe that these two apparently quite different mechanisms can be derived as special results of a more general case. The figure also illustrates an approach wherein the chord is drawn from a starting point a distance, s, from the center of the circle. It is intuitively obvious (and easy to prove) that if s equals r the probability in question is $1/3$ and that the limit of the probability as s approaches infinity is $1/2$.

It is of some interest to examine the general situation and probability distributions as s ranges from 0 to ∞. There are three separate cases.

For $s \leqq r/2$, it is clear that the probability is 1 that a chord so drawn shall be greater than $\sqrt{3}r$ (the length of the side of an inscribed equilateral triangle).

For $r/2 \leqq s \leqq r$, θ may be assumed uniformly distributed from 0 to 90°. The probability is then easily found to be $(2/\pi)$ arcsin $(r/2s)$ and this is equal to $1/3$ for $s=r$.

For $s \geqq r$, θ may be assumed uniformly distributed between 0 and arcsin (r/s) (the angle which yields the tangent). The probability is then found to be

$$\frac{\arcsin\,(r/2s)}{\arcsin\,(r/s)};$$

this is equal to $1/3$ for $r=s$ and increases monotonically to approach $1/2$ as a limit as $s \to \infty$.

A NEW METHOD OF SOLVING A QUARTIC

PAVAN K. MALHOTRA, South Dakota School of Mines and Technology

Let the quartic equation be written in the form:

$$x^4 + px^3 + qx^2 + rx + s = 0. \tag{1}$$

Break the coefficient on the x^2 term so that $qx^2 = (q-k)x^2 + kx^2$, where k is some constant whose value can be determined. Therefore (1) becomes

$$x^4 + px^3 + kx^2 + (q-k)x^2 + rx + s = 0. \tag{2}$$

Regroup the terms in (2) in the following manner:

$$x^4 + (px + k)x^2 + [(q-k)x^2 + rx + s] = 0. \tag{3}$$

Solve the above equation as a quadratic in x^2, treating $(px+k)$ and $[(q-k)x^2 + rx + s]$ as constant factors:

$$x^2 = \frac{-(px+k) \pm \sqrt{x^2(p^2 + 4k - 4q) + x(2pk - 4r) + k^2 - 4s}}{2}. \tag{4}$$

The value of k shall now be determined so that the quantity under the radical

in (4) is a perfect square, *i.e.*,

$$2(p^2 + 4k - 4q)^{1/2}(k^2 - 4s)^{1/2} = 2pk - 4r. \tag{5}$$

Squaring and simplifying (5) results in the following cubic equation in k with roots k_1, k_2, and k_3:

$$k^3 - qk^2 + (pr - 4s)k + 4qs - p^2s - r^2 = 0. \quad (k = k_1, k_2, k_3). \tag{6}$$

Putting any one of these values (k_1, k_2, k_3), *i.e.* k_i, into (4) results in

$$x^2 = [-px - k_i \pm (Lx + M)]/2, \tag{7}$$

where $L = (p^2 + 4k_i - 4q)^{1/2}$ and $M = (k_i^2 - 4s)^{1/2}$. But (7) is simply two separate quadratics in x, namely

$$x^2 + x(p - L)/2 + (k_i - M)/2 = 0, \tag{8}$$

$$x^2 + x(p + L)/2 + (k_i + M)/2 = 0. \tag{9}$$

Each of these quadratics can be solved giving the following four values for x which are obviously the roots of (1):

$$\begin{aligned}
x_1 &= \frac{(L - p)/2 + \sqrt{(L - p)^2/4 - 4(k_i - M)/2}}{2}, \\
x_2 &= \frac{(L - p)/2 - \sqrt{(L - p)^2/4 - 4(k_i - M)/2}}{2}, \\
x_3 &= \frac{-(L + p)/2 + \sqrt{(L + p)^2/4 - 4(k_i + M)/2}}{2}, \\
x_4 &= \frac{-(L + p)/2 - \sqrt{(L + p)^2/4 - 4(k_i + M)/2}}{2}.
\end{aligned} \tag{10}$$

Example. (All the roots are imaginary.)

$$x^4 - 10x^3 + 57x^2 - 148x + 200 = 0,$$

$$x^4 - 10x^3 + kx^2 + (57 - k)x^2 - 148x + 200 = 0,$$

$$x^4 - x^2(10x - k) + (57 - k)x^2 - 148x + 200 = 0.$$

$$\text{(i)} \quad x^2 = \frac{(10x - k) \pm \sqrt{100x^2 + k^2 - 20kx - 228x^2 + 4kx^2 + 592x - 800}}{2}$$

$$= \frac{(10x - k) \pm \sqrt{x^2(100 - 228 + 4k) - x(20k - 592) + k^2 - 800}}{2}$$

$$2\sqrt{4k - 128}\sqrt{k^2 - 800} = 20k - 592.$$

From the above we get the resolvent equation

$$k^3 - 57k^2 + 680k + 3696 = 0, \quad k_1 = 33, \quad k_2 = 28, \quad k_3 = -4.$$

Putting k_1 in (i)

$$x^2 = \frac{(10x - 33) \pm \sqrt{4x^2 - 68x + 289}}{2}$$

$$= \frac{(10x - 33) \pm (2x - 17)}{2} = \begin{cases} 6x - 25, \\ 4x - 8. \end{cases}$$

$$x^2 - 6x + 25 = 0, \qquad x^2 - 4x + 8 = 0,$$

$$x_1 = 3 + 4i, \qquad x_3 = 2 + 2i,$$

$$x_2 = 3 - 4i. \qquad x_4 = 2 - 2i.$$

INTEGRATION BY MATRIX INVERSION

WILLIAM SWARTZ, Montana State College

The integration of several functions using differential operators was considered by Osborn.* The integration of these and certain other functions by matrix inversion can furnish an application of several aspects of matrix theory of interest to the student of matrix algebra.

Let V be the vector space of differentiable functions. Let the n-tuple f be a basis spanning a subspace S of V which is closed under differentiation. Then differentiation comprises a linear transformation T of S into itself. If the matrix A represents T relative to f, then when A is nonsingular the elements of fA^{-1} yield antiderivatives of the elements of f.

To integrate $e^{ax} \sin bx$ and $e^{ax} \cos bx$ consider $f = (e^{ax} \sin bx,\ e^{ax} \cos bx)$. Then

$$fT = (ae^{ax} \sin bx + be^{ax} \cos bx,\ -be^{ax} \sin bx + ae^{ax} \cos bx)$$

and

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Furthermore

$$A^{-1} = \frac{1}{a^2 + b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

and then

$$fA^{-1} = \left(\frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx), \frac{e^{ax}}{a^2 + b^2} (b \sin bx + a \cos bx) \right)$$

yields antiderivatives of the elements of f.

* Roger Osborn, Note on integration by operators, this MONTHLY, vol. 64, 1957, p. 431.

To derive the formula

$$\int x^n e^x dx = e^x[x^n - nx^{n-1} + n(n-1)x^{n-2} - \cdots + (-1)^n n!] \tag{1}$$

for positive integers n consider $f=(e^x, xe^x, x^2e^x, \cdots, x^ne^x)$. Then

$$fT = (e^x, e^x + xe^x, \cdots, nx^{n-1}e^x + x^ne^x)$$

and there follows the interesting matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 0 & 1 & 2 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 & n-1 & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 & n \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 1 \end{pmatrix}$$

Since only an antiderivative of the last element of f is required, one is only interested in the last column of A^{-1}. Due to the peculiar form of A the inverse is easily deduced. One surmises that the last column of A^{-1} is the transpose of the row

$$\left((-1)^n n!, (-1)^{n-1} n!, (-1)^{n-2}\frac{n!}{2!}, \cdots, n(n-1), -n, 1\right). \tag{2}$$

That this supposition is correct may be verified by induction on n. In this connection it is useful to consider the $(n+2)$ rowed matrix corresponding to A as a partitioned matrix containing A as a principal submatrix. Finally one notes that multiplication of (2) by f yields the required formula (1).

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1311. *Proposed by G. K. Wenceslas, Santa Monica, Calif.*

The dissection of a unit square into one $1/8 \times 1$ and two $1/2 \times 7/8$ rectangular pieces shows that it is possible to cover the unit square with three sets each having diameter $d = \sqrt{65}/8$. Prove that the unit square cannot be covered by three sets all of which have diameter less than d.

E 1312. *Proposed by C. F. Pinzka, University of Cincinnati*

"What kind of doodling is that?" I asked, noticing that Willie had the following steps on his paper:

(0) x_1y_2

(1) $x_3y_4y_5x_6$

(2) $x_7y_8y_9x_{10}y_{11}x_{12}x_{13}y_{14}$

"It's sequential doodling," replied Willie. "I get each new step from the last step by inserting an x after each y, a y after each x, and continuing the subscripts in order."

"Does your doodling have any interesting properties?" I asked in jest.

"Oh, yes," said Willie. "In step n you will find that the sum of the kth powers of the x subscripts equals the sum of the kth powers of the y subscripts for $k = 0, 1, \cdots, n$. In fact, you can use any numbers in arithmetic progression for the subscripts and this property will still hold."

Prove that Willie's assertions hold for all (nonnegative integers) n.

E 1313. *Proposed by Roger Pinkham, Princeton University*

If $pz^2 - qz + 1 = 0$, then what values of p and q yield roots exterior to the unit circle?

E 1314. *Proposed by J. M. Gandhi, Jain Engineering College, Gurukul, Panchkoola, India*

Let $S(n, i) = \sum a_1 \cdots a_i$, where the sum is taken over all possible integral choices of the a's such that $1 \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_i \leqq n$, and put $S(0, i) = 0$, $S(n, 0) = 1$. Prove that for all positive integral n

$$\sum_{k=0}^{n-1} (-1)^k (n-k)! S(n-k, k) = 1.$$

E 1315. *Proposed by R. L. Caskey, Oklahoma State University*

Let M represent a magic square of order n, the elements of which are the positive integers from 1 to n^2. Let M_i represent the magic square obtained by replacing each element k of M by $k+(i-1)n^2$. Prove that the square array of elements obtained by replacing each element i of M by M_i is a magic square of order n^2.

SOLUTIONS

An Insolvable Diophantine Equation

E 1281 [1957, 592]. *Proposed by D. J. Newman, AVCO Research Division, Lawrence, Mass.*

Prove that no perfect square is 7 more than a perfect cube.

Solution by M. S. Klamkin, AVCO Research Division, Lawrence, Mass. This is a problem due to V. A. Lebesgue in 1869. A reference to this and the following proof are given in H. Davenport, *The Higher Arithmetic* (1952), p. 160.

If $y^2=x^3+7$, then x must be odd, since a number of the form $8k+7$ cannot be a square. Now

$$y^2 + 1 = (x + 2)(x^2 - 2x + 4) = (x + 2)[(x - 1)^2 + 3],$$

and the final factor is of the form $4n+3$, and hence must have a prime factor of this same form. But it is well known that y^2+1 cannot have a prime factor of this form.

Also solved by Merrill Barnebey, W. J. Blundon, Leonard Carlitz, P. L. Chessin, I. A. Dodes, N. J. Fine, Jose Gallego-Diaz, Joseph Hershenov and Joseph Muskat (jointly), Sam Kravitz, D. C. B. Marsh, C. S. Ogilvy, Anatol Rapoport, G. J. Simmons, Art Steger, Dmitri Thoro, R. M. Warten, and the proposer. Late solutions by D. R. Breach, D. A. Breault, Joe Lipman, and E. P. Starke.

L. E. Dickson, *History of the Theory of Numbers*, vol. 2, p. 534, gives the Lebesgue reference.

Triangle of Minimum Perimeter

E 1282 [1957, 592]. *Proposed by Leon Bankoff, Los Angeles, Calif.*

Tangents PQ, PR are drawn from a point P to a circle (O). Another tangent touches the circle at a point B on the minor arc QR and cuts PQ, PR at A, C. Show that QR is equal to the sum of the sides of the triangle of minimum perimeter that can be inscribed in triangle OCA.

Solution by C. F. Pinzka, University of Cincinnati. Let QR cut OA, OC at A', C'. Then $QA'=BA'$, $RC'=BC'$, and QR equals the perimeter of triangle $A'BC'$. But $A'BC'$ is the triangle of minimum perimeter, since it makes equal angles with the sides of acute angled triangle OCA.

Also solved by J. W. Armstrong, J. W. Clawson, A. L. Epstein, Jose Gallego-Diaz, J. D. E. Konhauser, D. C. B. Marsh, William Moser, C. S. Ogilvy, L. A. Ringenberg, Roscoe Woods, and the proposer. Late solution by Geoffrey Kandall.

A Sufficient Condition for a Real Polynomial to Have Some Imaginary Zeros

E 1283 [1957, 592]. *Proposed by J. W. Andrushkiw, Seton Hall University*

If three consecutive coefficients of the real polynomial

$$P(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$$

are equal and different from zero, show that the zeros of $P(x)$ are not all real.

I. *Solution by D. C. B. Marsh, Colorado School of Mines.* We assume, without loss of generality, that $P(0) \neq 0$. If the real polynomial $P(x)$ has at least three consecutive coefficients equal, then $Q(x) = (x-1)P(x)$ has at least two consecutive zero coefficients, and Descartes' rule of signs shows that the sequences of coefficients of $Q(x)$ and of $Q(-x)$ possess a total of no more than $n-1$ variations in sign. It follows that $Q(x)$ has no more than $n-1$ real zeros, and consequently $P(x)$ has no more than $n-2$ real zeros.

II. *Solution by R. E. Greenwood, University of Texas.* Oliver D. Kellogg, *Annals of Mathematics*, ser. 2, vol. 9 (1908), p. 97, proved that if all the coefficients of

$$f(x) = c_0 + c_1x + \cdots + c_nx^n$$

are real, and if all the zeros of $f(x)$ are real, then all the determinants

$$D_j = \begin{vmatrix} c_{j-1} & c_j \\ c_j & c_{j+1} \end{vmatrix}$$

are negative for $k < j < n$, where c_k is the first nonzero coefficient in $f(x)$.

The given problem is an immediate consequence of this result, for, since one $D_j = 0$, the polynomial must have some imaginary zeros.

Also solved by W. E. Deskins, A. L. Epstein, Jose Gallego-Diaz, F. D. Parker, J. J. Price, J. E. Scroggs, Marlow Sholander, and the proposer. Late solutions by Juris Hartmanis, E. P. Starke, and D. E. Thoro.

Editorial Note. If we assume $P(0) \neq 0$, then the restriction that the three equal coefficients be different from zero is not necessary.

Radius of Curvature of a Conic

E 1284 [1957, 592]. *Proposed by Charles Fox, McGill University*

Let P be any point on a central conic C, and let the normal to C at P and the polar of P with respect to the director circle of C meet at Q. Show that PQ is equal to the radius of curvature of C at P. What form does this result take for a parabola?

I. *Solution by Roscoe Woods, State University of Iowa.* Denote the center of C by O', the director circle of C by C', the polar of P with respect to C' by p, and the line through O' parallel to the tangent to C at P by L. Let d denote the distance from P to p, r the distance from L to P, R' the distance PQ, and d' the distance $O'P$. Each distance may be considered positive. From similar triangles it follows that $R' = dd'/r$. It is to be proved that $R' = R$, where R is the radius of curvature of C at P.

In rectangular coordinates (X, Y), let the equation of C be taken in the form $Y^2=aX^2+2bX$, $b\neq 0$, and a an arbitrary constant. The radius of curvature R of C at $P(x, y)$ is given by $R=z^{3/2}/b^2$, where

$$z = x^2(a^2 + a) + 2bx(a + 1) + b^2.$$

The following equations and relations may now be readily found:

$$C'\colon a(X^2 + Y^2) + 2bX + b^2 = 0,$$

$$p\colon a(xX + yY) + b(X + x) + b^2 = 0,$$

$$L\colon (aX + b)(ax + b) - ayY = 0,$$

$$d = z/u, \quad \text{where} \quad u = [(ax + b)^2 + a^2y^2]^{1/2},$$

$$r = (b^2/a)/z^{1/2} \quad \text{and} \quad d' = u/a.$$

By substitution we find $R'=dd'/r=R$.

If we let a approach 0, the conic C approaches the parabola $Y^2=2bX$ and R' approaches $(2x+b)^{3/2}/b^{1/2}$, the radius of curvature of the parabola at P. We also note that the director circle C' approaches the line $X=-b/2$, the directrix of the parabola, and p approaches the line $X=-b-x$. This line has the property that if Q is any point on it, the line segment PQ is bisected by the directrix. For a parabola, then, the result may be stated thus: If P is any point of a parabola and if the normal to the parabola at P cuts the directrix in S, PS is equal in length to half the radius of curvature of the parabola at P.

Remarks. (1) This problem is discussed in Jacob Steiner's *Gesammelte Werke*, vol. 2, pp. 341–342. (2) The result of this problem furnishes a simple construction for the center of curvature of a conic section. Numerous constructions for the center of curvature of a central conic may be found in a paper by F. Balitrand in *L'Enseignement Mathématique*, vol. 18, pp. 260–268.

II. *Solution by Jose Gallego-Diaz, Vanderbilt University.* Serret has shown (*Nouvelles Annales*, 1861, p. 80) that if a triangle T which is self-conjugate for a conic C approaches as a limit a point P on C, then the circumcircle of T approaches as a limit the circle Γ externally tangent to C at P and having diameter equal to the radius of curvature of C at P. But, by a theorem of Faure (see, *e.g.*, Salmon, *A Treatise on Conic Sections*, Ex. 2, p. 341), the circumcircle of a triangle T which is self-conjugate for a central conic (parabola) C is orthogonal to the director circle (directrix) of C. Consequently, the circle Γ is orthogonal to C'. But the circle on PQ as a diameter is orthogonal to C'. It follows that PQ is equal to the radius of curvature of C at P.

An interesting case is that in which C is a rectangular hyperbola, for it is known that in this case the circumcircle of a triangle T which is self-conjugate for C passes through the center O' of C (see Salmon, Ex. 5, p. 215). Point Q is thus easily located as the intersection of the normal at P with the line through O' perpendicular to $O'P$.

Also solved by Roland Deaux, Jose Gallego-Diaz (analytically), D. C. B. Marsh, Sister M. Stephanie, P. D. Thomas, and the proposer.

Bazin's Theorem

E 1285 [1957, 592]. *Proposed by James Ax and Lawrence Shepp, Polytechnic Institute of Brooklyn*

Let A and B be two nth order determinants, and let C be a third determinant whose (i, j)th element is the determinant A with its ith column replaced by the jth column of B. Show that $C = A^{n-1}B$.

Solution by D. A. Robinson, University of Wisconsin. Let a, b be the matrices whose determinants are A, B respectively, and let c be the matrix with determinant C described above. Let A_{ij} be the cofactor of the (i, j)th element of a and let adj a be the matrix whose (i, j)th element is A_{ji}. Then $c = (\text{adj } a)b$. Taking determinants and noting Cauchy's result that the determinant of adj a is A^{n-1}, the desired result follows.

It is interesting to note the following generalization. If k distinct columns of B should be substituted for k distinct columns of A in all possible ways and the resulting determinants arranged lexicographically to form C, then

$$C = A^{\binom{n-1}{k}} B^{\binom{n-1}{k-1}}.$$

This result and several others appear in Aitken, *Determinants and Matrices* (1956), pp. 101–103.

Also solved by J. F. Burke, A. L. Epstein, M. S. Klamkin, D. C. B. Marsh, Chih-yi Wang, David Zeitlin, and the proposers. Late solutions by D. R. Breach and Joe Lipman.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4783. *Proposed by Leonard Carlitz, Duke University*

Find the polynomial solution of the system

$$xf'(x) - g'(x) = \mu f(x),$$
$$xg'(x) - f'(x) = \nu g(x),$$

where μ, ν are assigned constants.

4784. *Proposed by D. W. Jonah, Purdue University*

In John L. Kelley, *General Topology*, p. 18, a definition of a ring is given in which the left and right distributive laws are replaced by the composite distributive law: $(u+v)(x+y)=ux+uy+vx+vy$.

(a) Show by an example that such a system is not necessarily a ring.

(b) Show that if such a system contains an element a such that $a0=0$ (in particular, if the system has a multiplicative identity), then the system is a ring.

4785. *Proposed by Paul Erdös, University of British Columbia*

Let $n_1<n_2<\cdots$ be a sequence of integers. Denote by $f(N)$ the number of the n's not exceeding N. Assume that $f(2N)-f(N)<c_1$. Then there is a constant c_2, depending only on c_1, such that as $x\to 1$

$$\left|\sum_{k=1}^{\infty} x^{n_k} - f\left(\frac{1}{1-x}\right)\right| < c_2.$$

4786. *Proposed by Ky Fan, Oak Ridge National Laboratory*

Let A, B be two positive definite Hermitian matrices of order n, and let $C=A+B$. For any positive integer $p<n$, let A_p denote the principal submatrix of A formed by the first p rows and columns, and let B_p, C_p have similar meaning. Prove

$$\left(\frac{\det C}{\det C_p}\right)^{1/(n-p)} \geqq \left(\frac{\det A}{\det A_p}\right)^{1/(n-p)} + \left(\frac{\det B}{\det B_p}\right)^{1/(n-p)}.$$

[The case $p=n-1$ is known and due to H. Bergström (See R. Bellman, this MONTHLY, vol. 62, 1955, pp. 172–173). Also the inequality of Minkowski (Hardy, Littlewood and Polya, *Inequalities*, 1934, p. 35) may be regarded as the case $p=0$.]

4787. *Proposed by M. S. Klamkin, AVCO Research Division, Lawrence, Mass.*

Express as a single definite integral

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{(m+n+r)!\, m^n n^r}{m!n!r!(r+1)^{m+n+r+1}}.$$

SOLUTIONS

Generators of Ideals

4732 [1957, 202]. *Proposed by D. J. Newman, AVCO Research Division, Lawrence, Mass.*

Show that, in the ring of polynomials with integral coefficients, $(J(x))$, there are ideals which require arbitrarily many generators.

Solution by the proposer. We prove $(2^n, 2^{n-1}x, \cdots, x^n) = I_n$ requires $n+1$ generators. For, consider I_n modulo I_{n+1}. We claim that this contains at least 2^{n+1} elements; that is, there are 2^{n+1} elements of I_n whose differences do not lie in I_{n+1}. If we form the 2^{n+1} elements

$$\sum_{i=0}^{n} a_i 2^{n-1} x^i, \qquad a_i = 0 \text{ or } 1,$$

then these are all incongruent. For if

$$\sum_{i=0}^{n} (a_i - b_i) 2^{n-i} x^i \in I_{n+1},$$

then, setting $x = 2y$,

$$\sum_{i=0}^{n} (a_i - b_i) y^i \in (2, 2y, 2y^2, \cdots, 2y^{n+1})$$

and so all the $a_i - b_i$ are even, whence $a_i = b_i$.

Now suppose $I_n = (P_1, P_2, \cdots, P_r)$. If $Q(x)$ is any polynomial such that $Q(0)$ is even, then $Q \cdot P_i \in I_{n+1}$, hence every element in $(P_1, \cdots, P_r)$ is congruent to

$$\sum_{i=1}^{r} \lambda_i P_i, \qquad \lambda_i = 0 \text{ or } 1,$$

but therefore the total number of elements in $(P_1, \cdots, P_r)$ modulo I_{n+1} is at most 2^r. Hence, finally, $2^r \geqq 2^{n+1}$, $r \geqq n+1$.

Dirichlet Series

4734 [1957, 276]. *Proposed by George Brauer, University of Minnesota*

Construct an ordinary Dirichlet series $F(s) = \sum_{n=1}^{\infty} c_n n^{-s}$ such that

$$\sum_{n=1}^{\infty} |c_n| < \infty \quad \text{and} \quad \lim_{\sigma \to 0+} |F'(\sigma + i\tau)| = \infty$$

for almost all points on the τ-axis ($s = \sigma + i\tau$). (A Taylor series with analogous properties is given in Rudin's paper, *On a problem of Bloch and Nevanlinna*, Proc. Amer. Math. Soc., vol. 6, 1955, pp. 202–204.)

Solution by the proposer. Take $n_1 = 1$; and having chosen $n_2, n_3, \cdots, n_{k-1}$, take a positive integer n_k such that

$$n_k > k^2 n_{k-1}, \tag{1}$$

$$\sum_{s=1}^{k-1} n_s [1 - (1 - n_k^{-1/2})^{n}] < 1. \tag{2}$$

Also let

$$f(z) = \sum_{k=1}^{\infty} k^{-2} z^{n_k}. \tag{3}$$

Rudin (in the paper already cited) shows that the series (3) converges absolutely on $|z| = 1$ while $f'(re^{i\theta}) \to \infty$ as $r \to 1-0$ for almost all values of θ. Let N_k denote the least integer greater than e^{n_k}, and let

$$F(s) = \sum_{k=1}^{\infty} k^{-2} N_k^{-s}, \qquad s = \sigma + i\tau. \tag{4}$$

The series (4) converges absolutely for $\sigma = 0$. We show that $F'(\sigma + i\tau) \to \infty$ as $\sigma \to 0^+$, for almost all values of τ.

Let $r = e^{-\sigma}$, $s = \sigma + i\tau$, then (differentiation is with respect to s)

$$\begin{aligned} &\left| -re^{-i\tau} f'(re^{-i\tau}) - F'(\sigma + i\tau) \right| \\ &\quad = \left| -\sum_{k=1}^{\infty} k^{-2} n_k e^{-n_k s} + \sum_{k=1}^{\infty} k^{-2} \log N_k e^{-s \log N_k} \right| \\ &\quad = \sum_{k=1}^{\infty} k^{-2} \left\{ \log N_k e^{-s \log N_k} - \log N_k e^{-n_k s} + \log N_k e^{-n_k s} - n_k e^{-n_k s} \right\} \\ &\quad \leqq \sum_{k=1}^{\infty} k^{-2} \left\{ \log N_k \left| e^{-s \log N_k} - e^{-n_k s} \right| + \left| \log N_k - n_k \right| \right\}. \end{aligned}$$

Since $\log (N_k - 1) \leqq n_k < \log N_k$, $0 < \log N_k - n_k \leqq 1/(N_k - 1)$ and

$$\left| e^{-s \log N_k} - e^{-n_k s} \right| \leqq |s| (\log N_k - n_k)$$

we have, for $0 < \sigma < 1$,

$$\begin{aligned} &\left| -re^{-i\tau} f'(re^{-i\tau}) - F'(\sigma + i\tau) \right| \\ &\quad \leqq (|s| + 1) \sum_{k=1}^{\infty} \log N_k / k^2 (N_k - 1) \\ &\quad \leqq (|\tau| + 2) \sum_{k=1}^{\infty} \log N_k / k^2 (N_k - 1) \leqq M(|\tau| + 2). \end{aligned}$$

As $\sigma \to 0^+$, $r \to 1-0$; hence $\lim_{(\sigma \to 0^+)} F'(\sigma + i\tau) = \infty$ for all values of τ such that $\lim_{(r \to 1-0)} re^{-i\tau} f'(re^{-i\tau}) = \infty$; that is $\lim_{(\sigma \to 0^+)} F'(\sigma + i\tau) = \infty$ for almost all values of τ.

The Encyclopaedias with Missing Volumes

4736 [1957, 277]. *Proposed by M. v. Ments, Jerusalem, Israel*

An encyclopaedia consists of N volumes, and in a certain town p persons originally possessed complete sets. But by some event, a lot of volumes of the various sets were destroyed at random, so that only n_i volumes remain in the possession of the ith person. If all the townspeople now pool their books, what

is the probability that M volumes (out of the total N) are available somewhere in town?

*Solution by the proposer.** Let A_i be the event that the ith volume is missing from the collection of all the volumes remaining, and let $p_i = pr(A_i)$ be the probability of the event A_i. Let $p_{ij} = pr(A_iA_j)$ be the probability of both A_i and A_j, $p_{ijk} = pr(A_iA_jA_k)$, and so on, $i, j, k = 1, 2, \cdots, N$ with $i<j<k< \cdots$. It is immediately evident that

$$p_i = \prod_{k=1}^{p} \frac{\binom{N-1}{n_k}}{\binom{N}{n_k}}, \quad p_{ij} = \prod_{k=1}^{p} \frac{\binom{N-2}{n_k}}{\binom{N}{n_k}}, \quad p_{ijk} = \prod_{k=1}^{p} \frac{\binom{N-3}{n_k}}{\binom{N}{n_k}}, \cdots.$$

Now define $S_1 = \sum p_i$, $S_2 = \sum p_{ij}$, $S_3 = \sum p_{ijk}$, *etc.* We will have

$$S_t = \binom{N}{t} \prod_{k=1}^{p} \binom{N-t}{n_k} \Big/ \binom{N}{n_k}, \qquad t = 1, 2, \cdots, N.$$

Fréchet has proved that, for any integer r with $1 \leqq r \leqq N$, the probability $P_{[r]}$ that exactly r among the N events $A_1, \cdots, A_N$ occur simultaneously is given by

$$(1) \qquad P_{[r]} = S_r - \binom{r+1}{r} S_{r+1} + \binom{r+2}{r} S_{r+2} - + \cdots + \binom{N}{r} S_N.$$

(See Feller, *Probability Theory and its Applications*, 1950, pp. 64–65.)

Now the probability that exactly M of the volumes remain is the probability that exactly $r = N - M$ of the events $A_1, A_2, \cdots$ occur. According to (1) the desired probability is then

$$P_{[N-M]} = S_{N-M} - \binom{N-M+1}{N-M} S_{N-M+1} + \binom{N-M+2}{N-M} S_{N-M+2} - + \cdots$$
$$\pm \binom{N}{N-M} S_N.$$

Upon substituting the value of S_t from above and employing the easy identity

$$\binom{N-M+v}{N-M}\binom{N}{N-M+v} = \binom{N}{M}\binom{M}{v},$$

this becomes

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* Abridged through the use of a theorem of Fréchet.—*Ed.*

$$\frac{\binom{N}{M}}{\prod_{k=1}^{p}\binom{N}{n_k}} \sum_{v=0}^{\infty} (-1)^v \binom{M}{v} \prod_{k=1}^{p} \binom{M-v}{n_k}.$$

Cesáro First Order Mean

4742 [1957, 370]. *Proposed by Joshua Barlaz, Rutgers University*

Evaluate the Cesáro first order mean for the series $\sum_{n=2}^{\infty} (-1)^n \log n$.

Solution by Robert Breusch, Amherst College. The partial sums are

$$S_{2n} = \log \frac{2 \cdot 4 \cdot \cdots \cdot (2n)}{3 \cdot 5 \cdot \cdots \cdot (2n-1)}, \qquad S_{2n+1} = \log \frac{2 \cdot 4 \cdot \cdots \cdot (2n)}{3 \cdot 5 \cdot \cdots \cdot (2n+1)}.$$

Thus

$$S_{2n} + S_{2n+1} = \log \frac{2 \cdot 2}{1 \cdot 3} \frac{4 \cdot 4}{3 \cdot 5} \cdots \frac{(2n)(2n)}{(2n-1)(2n+1)} = \text{lo } (\pi/2) + O(1/n).$$

Therefore

$$\sum_{n=2}^{N} S_n = \frac{N}{2} \log (\pi/2) + O(\log N), \qquad \lim_{N \to \infty} \frac{1}{N} \sum_{n=2}^{N} S_n = \frac{1}{2} \log (\pi/2).$$

Also solved by J. L. Alperin, R. G. Buschman, N. J. Fine, and Chih-yi Wang.

Editorial Note. As Buschman notes, the result follows directly from G. H. Hardy, *Divergent Series*, p. 347, lines 6 to 10.

A Sequence Developed from a Recurrence Relation

4743 [1957, 436]. *Proposed by Aaron Herschfeld, Canisius College, Buffalo, N. Y.*

Given an arbitrary positive number u_0, consider the sequence u_n, where $u_n = u_{n-1} + 1/u_{n-1}$. Prove the relation $u_n = \sqrt{2n} + (\log n)/\sqrt{32n} + O(1/\sqrt{n})$.

Solution by Leopold Flatto, Reeves Instrument Corporation, New York City. From the given recurrence relation we get $u_n^2 = u_{n-1}^2 + 1/u_{n-1}^2 + 2$ and hence

$$u_n^2 = u_0^2 + \left(\frac{1}{u_0^2} + \frac{1}{u_1^2} + \cdots + \frac{1}{u_{n-1}^2}\right) + 2n, \tag{1}$$

$$u_n^2 > 2n. \tag{2}$$

Using (1) and (2) we have

$$u_n^2 - 2n < (u_0^2 + 1/u_0^2) + \frac{1}{2} \sum_{k=1}^{n-1} 1/k < \left(u_0^2 + 1/u_0^2 + \frac{1}{2}\right) + \frac{1}{2} \log n,$$

and so

$$(3)\qquad \sum_{n=1}^{\infty}\left(\frac{1}{2n}-\frac{1}{u_n^2}\right)=\sum_{n=1}^{\infty}\frac{u_n^2-2n}{2nu_n^2}<\sum_{n=1}^{\infty}\frac{u_0^2+1/u_0^2+\frac{1}{2}+\frac{1}{4}\log n}{4n^2}.$$

Hence (1) may be rewritten as

$$u_n^2=2n+\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2(n-1)}\right)+O(1)=2n+\frac{1}{2}\log n+O(1).$$

Then finally we have $u_n=(2n+\frac{1}{2}\log n+O(1))^{1/2}=\sqrt{2n}(1+(\log n)/4n+O(1/n))^{1/2}=\sqrt{2n}(1+(\log n)/8n+O(1/n))$, from which the desired relation follows immediately. The positive square root is chosen since $u_0>0$ implies all $u_n>0$.

Also solved by L. R. Bragg, Robert Breusch, Emil Grosswald, Ya'akov Shima, Chih-yi Wang, G. S. Wells, and the proposer.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

The Theory of Lie Derivatives and Its Applications. By Kentaro Yano. Interscience, New York, 1957. x+299 pp. $8.75.

One of the remarkable results in classical differential geometry is the fact that a surface admits either no or a one-parameter or a three-parameter group of transformations; in the last case it is of constant curvature. In the present book one finds far-reaching generalizations of this result to Riemannian, affinely-connected, projectively-connected, conformally-connected spaces by Egorov, Muto, Vranceanu, Wang, and the author.

Naturally, a basic analytic tool is the Lie derivative. It is an operator defined by a vector field and acts on an arbitrary tensor field, or, more generally, on a geometric object. The vector field generates a one-parameter group of transformations and the Lie derivative is, roughly speaking, the derivative along its orbits. The notion is extremely natural and was used by Lie with success in various special cases. Its general formulation was due to W. Slebodzinski, according to the author (who is, incidentally, one of the leading experts on modern literature on differential geometry).

The book has ten chapters and an appendix. The first three chapters deal with generalities, the next five treat mostly the local theory of transformation groups in spaces which are of interest in differential geometry, and the last two chapters and the appendix study some global aspects of the problem. The latter

forms an introduction to a new field which should be full of promise. In fact, spaces with a transformation group present, in a sense, a certain amount of symmetry and are naturally more important than the general ones. The differential-geometric theory should be a worthy sequel to the theory of topological transformation groups.

The book is well written and is undoubtedly a valuable addition to mathematical literature.

S. CHERN
University of Chicago

Introduction to Abstract Algebra. By C. Racine. Viswanathan, Madras, 1957. xii+194 pp. 6 rupees.

The aim of this book, according to the author's preface, is "to make it easy for students to familiarize themselves with the essential elements of a fundamental structure of Modern Mathematics—to make it cheap too!" The result is a fairly tasteful selection of basic topics from modern algebra. The first three chapters of the book, dealing with groups, rings and fields, cover about the same ground as Volume 1 of van der Waerden's *Moderne Algebra*. Chapter four—the final chapter—treats matrices and their canonical forms.

The author has done a remarkably poor job of proofreading his manuscript. In the errata 65 items are listed, but there are perhaps a hundred more slips of a typographical or careless nature. For example, on p. 70, one "learns" that a sufficient condition for a ring to be a unique factorization domain is that it be Noetherian. But this is followed by the contradictory example $Z[\sqrt{-5}]$.

In the reviewer's opinion, Professor Racine has not achieved his difficult objective of writing a useful book for students. This "introduction" to algebra, written as it is in the unpalatable "definition-theorem-proof" jargon of a research paper, demands more mathematical maturity than can be expected of even a good graduate student. At best, the book can serve as a convenient set of lecture notes, to be used by graduate students cramming for an examination, or by an overburdened teacher following the geodesic of least resistance.

R. S. PIERCE
University of Washington

The Number-System. By H. A. Thurston. Interscience, New York, 1956. vii+134 pp. $2.50.

Every serious student of mathematics is exposed at some point in his education to the systematic construction of the complex numbers starting from the Peano postulates. This exposure may take place piecemeal. Construction of the integers and rational numbers is usually done in courses in abstract algebra; the passage from the rationals to the reals in courses in analysis; and the introduction of the complex numbers in courses in college algebra, or complex variables. For students who prefer to see the entire construction of the number-system in a single consecutive development the author has provided a most

acceptable alternative to the classical treatment by Landau in *Grundlagen der Analysis*. (Leipzig, 1930. English Translation, New York, 1951.)

The purpose and scope of the present book is the same as that of Landau's *Grundlagen* but the flavor is different. In the first place it has been made more palatable to the inexperienced reader by the inclusion of a preliminary, informal, heuristic discussion designed to motivate the postulates and definitions and generally to explain the formal treatment that follows. This informal discussion occupies a little more than one-third of the book and its style is both lucid and stimulating.

The formal treatment in Part 2 is, fittingly enough, in the severe mathematical style of "Definition:—Theorem:—Proof:" but again it is lucid and easily read. The order of presentation differs from that of Landau. The natural numbers are first imbedded in the integers which are then imbedded in the rational numbers. This order, though it no doubt departs from the historical, makes it possible to use the techniques of abstract algebra to greater advantage in the successive constructions. It also serves to separate as far as possible the algebraic part of the construction from the analytic, so that the first eight chapters of Part 2 contain the complete construction of the integers and of the rational numbers, constructions that never appear, as such, in Landau. The real numbers are defined by means of Cauchy sequences, the results being obtained for any ordered field. A discussion of the exponential function and the decimal representation of real numbers are included. Exercises are provided at the end of each chapter and these include several important results not formally stated in the text. "A Key to the Exercises" outlines the solutions. There is a short but adequate bibliography and an index.

D. C. MURDOCH
University of British Columbia

Understanding Arithmetic. By Robert L. Swain. Rinehart, New York, 1957. xxi+264 pp. $4.75.

This book deals with arithmetic on a fairly mature level and is addressed primarily to future teachers of arithmetic in the elementary schools. The following summary of its contents will indicate its scope.

The first chapter is historical in nature and discusses the Babylonian, Egyptian, Greek and Roman systems of numerals and compares them with the Hindu-Arabic system. Chapter 2 deals with sets, one to one correspondence and (finite) cardinal numbers. In Chapter 3 addition and multiplication are defined in terms of the cardinal numbers of the union and Cartesian product of disjoint sets. The "laws of arithmetic" are then derived and the inverse operations, subtraction and division, are introduced. The integers are defined by a comparatively concrete and colorful (no pun intended) version of the usual ordered pairs of natural numbers. The next three chapters are devoted to the algorisms for addition, subtraction, multiplication and division in the Hindu-Arabic system, first with base 10 and then in the duodecimal and binary scales.

Chapter 7 contains a little elementary number theory, including the proof of the existence of an infinite number of primes and of the unique factorization theorem. Fractions, decimals and percent are discussed in Chapters 8 and 9. Chapter 10 treats the rational number system, the existence of irrationals, the real numbers (informally, as infinite decimals), and the rational numbers as repeating decimals. The last two chapters discuss measurement and units, accuracy of approximations, significant digits, *etc.*

The book is written for the student with little or no background of college mathematics. The exposition is clear and the style is pleasing and likely to maintain the student's interest. There are frequent historical anecdotes and many useful references to books where the student can find more complete discussion of various mathematical topics. It is a pity that these references were not all listed again at the end of the book as a composite list of suggestions for further reading.

The author's remark that the book could well have been titled *What Every Teacher Ought to Know about Arithmetic* represents a modest enough aim and yet one that, if attained, would probably mean a significant educational advance.

D. C. MURDOCH
University of British Columbia

Vector Analysis. By Louis Brand. Wiley, New York, 1957. xiv+282 pp. $6.00.

This short work is a contribution to the field of elementary vectors. The treatment is concise, definite and vigorous. The first five chapters are concerned with the algebra and calculus of vectors. Vectors need something to develop with. For them geometry is natural and these chapters abound with numerous applications to this field. The other four chapters are brief treatments of Dynamics, Fluid Mechanics, Electrodynamics and Vector Spaces.

Chapter I. Vector Algebra. The topics are standard. The treatment is clear. The applications are on analytics, mechanics, plane and spherical trigonometry, and several of the theorems from projective geometry and college geometry.

Chapter II. Line Vectors. Applications to Statics.

Chapter III. Vector Functions of One Variable. Differentiation of vectors with applications to differential geometry and kinematics.

Chapter IV. Differential Invariants. Here one finds applications to surfaces and fields. Divergence and rotation are presented as invariants of a dyadic. The chapter ends with curvilinear coordinates.

Chapter V. Integral Theorems. The topics are the usual ones, line integrals, Green's Theorem, Stokes' Theorem, Divergence Theorem. The treatment is rather extensive. The best applications are in later chapters.

Chapter VI, . . . , IX. These are brief compact courses on the topics mentioned above. Perhaps they should be left for full courses in these topics.

The book would be suitable as a text for students past the calculus. The

average student would probably find the problems interesting, but difficult, at least until he acquires some skill in applying vector techniques.

EARL LA FON
University of Oklahoma

Models of Man. By Herbert A. Simon. Wiley, New York, 1957. xiv+287 pp. $5.00.

How would you describe the social behavior of man? The many aspects of his rational behavior? The author has selected from journal publications during the past decade sixteen of his essays (two co-authored) which analyze various facets of man's behavior. The papers are grouped into four parts; an introduction to each part links together the essays of that group and discusses their interconnection with the remaining parts. Some of the topics treated are: causal relations between variables; measurement of political power or influence; effect of predicting an election on changing its outcome; friendliness in small groups; pressures toward uniformity in social groups; why people join organizations, remain in or leave organizations; servomechanisms controlling production; the nature of rational choice. In the latter area the principle of bounded rationality is enunciated and other explanations of the choice process are attacked. Man's goal in decision making is not necessarily to maximize his return or benefit but rather to satisfice, to find a course of action that is good enough. The statement of the principle of bounded rationality is: "The capacity of the human mind for formulating and solving complex problems is very small compared with the size of the problems whose solution is required for objectively rational behavior in the real world—or even for a reasonable approximation to such objective rationality."

The physical sciences have long utilized the tools of mathematics. Mathematical formulation often assists in understanding scientific principles, makes further discovery easier and, in return, the sciences have challenged mathematics to develop more techniques. We are experiencing the early days of the era in which the social sciences will similarly be joining forces with mathematics for mutual benefit. "Models of Man" shows some of the mathematical paths which may be taken in economics, sociology, political science, psychology. Traveling the routes is pioneering; the future will tell which lead to significant progress, which are merely gropings. But always our sights must be forward. The reader must not expect to find this book a definitive treatise. Many of the models need broader and deeper development. Much new mathematics and strange approaches to old mathematics may help to handle the analyses. With an open mind, the reader can glimpse in this book fascinating vistas for future mathematical applications.

Some of the present mathematical topics which are utilized in the essays are set theory, probability theory, systems of differential equations, difference equations, and even the Brouwer fixed point theorem. Sometimes the down-to-earth usage of mathematics seems rather superficial; other times the author contends

that the mathematical formulation permitted the discovery of results not otherwise recognized. The reader interested in the role of mathematics in social science should certainly contemplate the first paragraph on page 89, the two paragraphs at the bottom of page 97, and most of page 142. Two sentences may be quoted here: "We must not expect to find the models we need ready-made in a mathematical textbook. If we are lucky, we shall not have to invent new mathematics, but we are very likely to have to assemble our model from a variety of raw materials." After you have decided how you would describe man's behavior in social and rational situations, then critically read these essays and compare their models with yours.

R. A. Good
University of Maryland

Wahrscheinlichkeits-Theorie. By Hans Richter. Springer (Die Grundlehren Der Mathematischen Wissenschaften in Einzeldarstellungen, Band LXXXVI). 435 pp. DM 69.60.

The book under review is a lucid and timely introduction to the modern theory of mathematical probability, and is warmly recommended to graduate students and all others who would like to attain the research level in this vital and constantly expanding field.

Recognizing that not only a rigorous development of the theory, but also the very working tools, depend upon the language and techniques of set theory, the author begins the volume with a very readable description of the fundamentals of set theory and measure theory.

He then turns to an interesting discussion of the meaning of probability, as a philosophic concept, as a working scientific concept, and finally as a precise mathematical concept.

The next section of the book is devoted to developing the basic results of probability theory from the fundamental axioms. This discussion prepares the way for the more advanced results which follow, and affords an opportunity for the treatment of many interesting and important problems.

Since the subsequent results require the Lebesgue-Stieltjes integral, the author once again detours to furnish an excellent introduction to integration theory. With this disposed of, there follow two long chapters devoted to the anatomy of random variables, characteristic functions, moments, and so forth.

The last chapter is dedicated to the study of convergence of sequences of random variables, the weak and strong versions of the law of large numbers and the central limit theorem.

An excellent feature of the book is a collection of problems at the ends of chapters with answers at the back of the book. The typography is attractive and readable, as is to be expected from Springer.

Richard Bellman
The RAND Corporation

Vorlesungen ueber Inhalt, Oberflaeche und Isoperimetrie. By H. Hadwiger. Springer Verlag, Berlin-Goettingen-Heidelberg, 1957. 312 pp. $11.95.

In this monograph the well-known author presents a formal development of the theory of measure. The early chapters are devoted to the discussion of an elementary measure space (P, Φ) consisting of the class of polyhedra P and a functional Φ defined on polyhedra in the usual way except that only translation-invariance is required. This concept is generalized in chapter III insofar as a measure space (F, φ) is defined as follows: F is a class of bounded point sets in the E_k which is closed under translation, additive, and contains the unit cube and the measure φ is translation-invariant, additive, nonnegative definite, and normalized. The elementary measure space as introduced in the beginning appears as embedded in (F, φ) if the latter has the measure theoretic field property. Jordan's measure space appears as the smallest measure space with the field property whereby "smallest" shall mean that it belongs to the class and is embraced by all other elements of the class. Lebesgue's measure space is characterized as the smallest complete measure space whereby "complete" has the usual meaning, *i.e.*, every set which can be enclosed by two measurable sets with arbitrarily small volume difference shall be measurable. Due to the author's approach in working with translation-invariance only and not with invariance with respect to the euclidean group, Tarski's measure space can be introduced for all $k \geqq 1$.

It is not possible to go into any details in discussing the rich content of the following chapters in the limited available space. The reader, however, may get a rough idea of what to expect by reading the remaining chapter headings in free translation: IV. Selected Studies in Set Geometry, V. Volume, Surface Area and Isoperimetry, VI. Convex Bodies and General Integral Geometry.

It should be stated that the present book is carefully planned and logically designed; it is written in an extremely clear style. It is of great advantage that every chapter is followed by an extensive bibliography.

The usual care and high quality of Springer Verlag is again manifested in this book.

HANS SAGAN
University of Idaho

Linear Algebra for Undergraduates. By D. C. Murdoch. Wiley, New York, 1957. xi+239 pp. $5.50.

The style of presentation here is well above average—direct, consistent, well-organized, succinct. The clarity of writing and publication should be welcome to instructors and especially to self-help students.

The matter of content presents some questions. With the author, I consider a course in vectors and matrices an appropriate undergraduate study. As he points out, it has in addition to ready application the advantage of a logical deductive development. I think the title "linear algebra" a bit ambitious for

such a course, especially when " . . . every effort has been made to keep the treatment of the various topics elementary and concrete and to introduce as few abstract ideas as possible." As to the avoidance of abstraction it may indeed make the "how" simple, but sooner or later the student is going to be stumped by the "why," and a book of this type will give him little help. Not only is there no help here from the point of view of fitting matrix algebra into the general picture as "*an* algebra" and the study of invariants under certain sets of transformations as "geometries," but there isn't even any practical motivation in the problems, which for the most part serve to round out the theory in the brief text. Such "applied" problems as appear are given in the sifted collected general form derived long after the construction of the mathematical model.

Considerable energy is going now into the development of freshman courses that will include some abstract ideas to organize burgeoning mathematical thought into a manageable form. In this light a deliberately nonabstract book, intended as a sequel to college algebra, seems a step backward.

MARGARET W. MAXFIELD
U. S. Naval Ordnance Test Station

BRIEF MENTION

The Tao of Science. By R. G. H. Siu. The Technology Press (MIT), and Wiley, New York, 1958. xvi+180 pp. $4.25.

This interesting account of the difference in basic philosophy between the Oriental and occidental teachings, written by a scientist (Ph.D. in Bio-organic Chemistry) who has received part of his training in the Orient and part in the United States, contains many points of interest.

Analytical Conics. By Barry Spain. Pergamon, New York, 1957. ix+145 pp. $5.00.

It is interesting to note a new book on the analytical geometry of the conic sections appearing at a time when most American universities are de-emphasizing conics. In addition to the usual analytic geometry of conic sections, the invariant theory classification of the general second degree equation and the concept of the line at infinity are introduced along with the basic ideas of cross ratio, homographic correspondence and line coordinates.

Probability Statistics and Truth. By Richard von Mises. Macmillan, New York, 1957. viii+244 pp. $5.00.

Whereas the original English translation of this book was based on the second German edition, this translation is based on the third (1951) German edition which includes work by Tornier, Doob, and others. The present translation by von Mises's widow, Professor Hilda Geiringer, is based on the earlier (1939) translation by Neyman, Scholl, and Rabinowitsch, supplemented by the new material in the 1951 edition. This translation will be doubly welcomed by mathe-

maticians since the earlier translation has been out of print for several years.

The Michigan Mathematical Journal. Vol. 4, No. 2. November 22, 1957. The University of Michigan Press, Ann Arbor. $1.00 per issue on direct orders from individuals, and $3.00 on orders from institutions.

It is a pleasure to call the *Michigan Mathematical Journal* to the attention of our readers. Its current editorial board consists of: Lipman Bers, Wilfred Kaplan, Irving Kaplansky, Edwin E. Moise, George Y. Rainich, Raymond L. Wilder; under the managing editorship of George Piranian.

Economic Models. By E. F. Beach. Wiley, New York, 1957. xi+227 pp. $7.50.

This book, written by an economist, contains what he considers "an elementary exposition of the mathematical and statistical implications of multiple relations in economic theory." It is intended to "fill the gap . . . between workers (in economics) who have a good grounding in advanced mathematics, and those whose mathematical background is sketchy." This reviewer is pleased to see a book with this purpose, but feels that the gap is considerably wider than the contents of this book can bridge. Only very elementary algebra is assumed, and the exposition falls considerably short of the calculus usually taught in integrated freshman mathematics courses. Current economic theory, as illustrated for example by the work of Ashby and Spivey, makes much use of linear difference equations, matric algebra and partial differential equations. However, Professor Beach's book should be welcomed as a firm step in the right direction.

Yano's Tables of Calculation. By Tsuneta Yano. The Tsuneta Yano Memorial Society, Tokyo, Japan, 1957. vi+162 pp.

The tables in this book are not as extensive or as carefully done as those of the standard American handbooks, but this is the only table, to my knowledge, which gives extensive conversion tables of weights and measures including the Oriental terms. It may prove a valuable reference for this reason.

Thermal Stresses. By B. E. Gatewood. McGraw-Hill, New York, 1957. xv+232 pp. $7.50.

"The mathematical procedures used in the first eight chapters do not go beyond elementary partial differential equations. Chapter 9 requires some knowledge of the theory of complex variables and of energy methods as used in structural analysis." Since the problems of high temperature are becoming more important in aircraft and reactor design, mathematicians may wish an up-to-date reference.

Approximations for Digital Computers. By Cecil Hastings, Jr. Princeton University Press, 1955. viii+201 pp. $4.00.

Although not a new book, it has apparently been overlooked in these re-

views. This monograph will prove of considerable use to the many mathematicians currently interested in high speed computing. Many of the approximation methods discussed here are not to be found conveniently in other sources.

CORRECTION

The publishers of *Introduction à l'Étude de l'Analyse Symbolique* (reviewed in this MONTHLY, vol. 65, 1958, pp. 215–216) have asked that the following addition and correction be made to the review.

The book contains 49 figures and the catalog price is 3,500 frs. plus postage.

NEWS AND NOTICES

EDITED BY L. J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

NATIONAL SCIENCE FOUNDATION SUMMER INSTITUTES

The National Science Foundation has announced the following Summer Institutes for college and high school teachers of mathematics. Inquiries about a particular institute should be sent to the Director listed below. Unless otherwise indicated, the Director is located at the same institution as the institute:

University of Kansas, June 9–August 2: Senior high school and college teachers. G. B. Price.

High school teachers only:

Carleton College, June 16–July 25: K. O. May.

Oberlin College, June 16–August 8: Wade Ellis

Rutgers University, June 29–August 8: E. P. Starke.

University of Buffalo, July 7–August 1: Harriet F. Montague.

University of Chicago, June 23–August 1: A. L. Putnam.

University of Notre Dame, June 19–August 6: A. E. Ross.

University of Vermont, June 23–August 8: N. J. Schoonmaker.

University of Wisconsin, June 30–August 22: For teachers of eleventh and twelfth grade mathematics. H. Van Engen.

University of Wyoming, June 16–August 8: W. Norman Smith.

Western Michigan University, June 23–August 1: C. H. Butler.

SUMMER SESSIONS

The following institutions announce advanced courses in mathematics for the summer of 1958.

Boston University, July 23 to August 14: Professor Scheid, seminar in applied mathematics.

Columbia University, Teachers College, July 7 to August 15: Professor Fehr, arithmetic in elementary school, history of mathematics; Professor Rosskopf, logic for teachers, current problems in the teaching of mathematics, departmental seminar in teaching mathematics; Dr. J. Hlavaty, probability and statistical inference for teachers, profes-

sionalized subject matter and advanced secondary school mathematics; Dr. Max Peters, the teaching of algebra, teaching of mathematics to exceptional children.

DePaul University, June 23 to August 1: Professor DeCicco, infinite series, linear algebras; Professor Merkes, college geometry; June 12 to August 1 (evenings): Professor Caton, mathematical statistics.

Duke University, June 10 to July 16: Professor Warner, finite mathematics; Professor Carlitz, advanced calculus. June 30 to August 8: Professor McGavock, fundamental concepts in algebra and geometry, finite mathematics. NSF Institute, June 9 to June 28: Professors Dressel, Elliott and McGavock, mathematics for teachers.

Kent State University, June 16 to July 19: Professor Brooks, selected topics for classroom teachers, Boolean algebra; Professor Olson, theory of numbers. July 21 to August 23: Professor Johnson, college geometry; Professor Kaiser, functions of a complex variable.

Michigan State University, June 24 to August 1: Professor Ewald, theory of numbers; Professor Grove, projective geometry; Professor Herzog, advanced calculus, theory of sets; Professor L. Kelly, concepts in geometry, mathematical problems; Professor Powell, concepts in calculus; Professor Sheedy, differential equations; Professor Weeg, computer coding; Visiting Professor H. J. Zassenhaus, representations of groups and Lie algebras. June 24 to August 22: Professor Campbell, matrices and groups, advanced mathematics for engineers I; Professor Hall, boundary value problems; Professor Helms, vector analysis, partial differential equations, advanced mathematics for engineers II; Professor Hill, functions of a real variable III; Professor Oehmke, advanced calculus III, functions of a complex variable I; Professor Reid, fluid dynamics, advanced mathematics for engineers III; Professor Wasserman, differential equations, advanced calculus II, numerical analysis I. NSF Institute for Community-Junior College Science and Mathematics Teachers, June 23 to August 1.

New York University, June 16 to July 24: Professor Shapiro, advanced calculus and applications I; Professor A. Peters, singular integral equations; Professor Magnus, special functions of mathematical physics; Professor Schwartz, introduction to Banach algebra. July 28 to September 5: Professor Rubin, advanced calculus and applications II; Professor Isaacson, elementary numerical methods and applications; Professor Shapiro, special topics in number theory (all half courses).

Northwestern University, June 21 to August 16: Engineering mathematics I; engineering mathematics II; numerical methods in mathematics; probability; history of mathematics; complex variables for applications; topics in modern mathematics for teachers; introduction to topology.

Syracuse University, June 30 to August 8: Research-oriented courses: Professor Kibbey, functions of a complex variable; Professor Gilbert, introduction to topology; Professor Gilchrist, programming digital computers. Courses for college teachers: Professor Gelbart, analysis and applications; Professor Exner, statistics. Courses for secondary teachers: Professor Moredock, the new Illinois-Beberman curriculum (demonstration class).

University of Chicago, June 23 to August 29: Professor Schilling, algebra II (vectors and matrices); Professor Chern, algebra IV (group theory), algebraic topology I; Professor Baily, point set topology, several complex variables; Professor Graves, calculus of variations; Visiting Professor Weil, automorphic functions of several variables; Visiting Professor Akizuki, local rings (July only). During July there will be a special program in algebraic geometry and several complex variables, with appropriate seminars.

University of Colorado, June 13 to August 23: Professor Bunt (University of Utrecht), teaching of secondary school mathematics, mathematics workshop in teaching problems; Professor Camp (University of Nebraska), mathematical statistics; Professor Nering (University of Arizona), game theory; Professor Magnus, theory of numbers, college geometry; Professor Briggs, functions of a complex variable.

University of Delaware, June 23 to August 1: Professor Wisner, fundamental concepts in mathematics; Professor Remage, introductory topology.

University of Florida, June 13 to August 9: Professor Ackerson, introduction to mathematical thought; Professor Meyer, mathematical statistics; Professor Sarafyan, higher mathematics for engineers and physicists; Professor Sobczyk, advanced topics in calculus; Professor Gaddum, introduction to topology; Professor Butson, vector analysis; Professor Moore, theory of groups of finite order; Professor Hutcherson, synthetic projective geometry; Professor Morse, foundation of geometry; Professor Kokomoor, history of elementary mathematics.

University of Michigan, June 23 to August 16: Professor Carver, calculus of finite differences; Professor Clarke, mathematical theory of probability, theory of statistics I; Professor Coburn, vector analysis; Professor Conner, advanced mathematics for engineers; Professor Craig, statistical analysis I; Professor Dushnik, introduction to differential equations, Fourier series and applications; Professor Dwyer, statistical analysis II, theory of statistics II; Dr. Galler, methods in high-speed computation; Dr. Griffin, advanced calculus; Professor Harary, introduction to matrices, introduction to the foundations of mathematics; Professor Hay, intermediate course in differential equations; Professor Leisenring, noneuclidean geometry, history of geometry; Dr. Maxwell, introduction to differential equations; Professor McLaughlin, theory of equations and determinants, algebra I; Professor Nesbitt, introduction to differential equations, mathematics of life insurance; Professor Ritt, introduction to functions of complex variable with applications, topics in modern mathematics for teachers; Professor Titus, modern operational mathematics, differential geometry; Professor Ullman, theory of functions of a complex variable; Professor Wilder, general spaces.

University of Minnesota, College of Science, Literature, and the Arts, June 16 to July 19: Professor Gelbaum, critical reasoning in mathematical analysis I, Laplace transforms; Professor Storvick, differential equations. July 21 to August 23: Professor Gelbaum, critical reasoning in mathematical analysis II; Professor Gil de Lamadrid, advanced algebraic theory, intermediate differential equations. It is expected that an Institute for High School Teachers of Mathematics will be conducted from June 16 to August 9.

University of Nebraska: Professor Basoco, differential equations; Professor Abel, topics in geometry; Professor Guy, mathematical analysis; Professor Leavitt, topics in algebra; Professor Jackson, topics in analysis. NSF Institute for High School Teachers: modern geometry, modern algebra.

University of North Carolina, June 3 to July 12: Professor Garner, history of mathematics; Professor Cameron, fundamental concepts; Professor Mackie, theory of equations; Professor Linker, differential equations; Professor Brauer, introduction to the theory of determinants and matrices; Professor Rohrbach, introduction into numerical analysis; Professor Pettis, topics in analysis, functions of a complex variable I. July 12 to August 20: Professor Hill, elementary mathematical statistics; Professor Lasley, analytic geometry from a higher standpoint; Professor MacNerney, functions of a complex variable II, summability; Professor Jones, foundations of geometry; Professor Wall, the theory of matrices.

University of Oklahoma, June 14 to August 12: Professor Bernhart, college geometry, vector analysis; Professor Giever, elementary differential equations; Professor Springer, partial differential equations; Professor Brixey, theory of numbers; Professor Goffman, topics in infinite series.

University of Pennsylvania, June 30 to August 9: introduction to real analysis, introduction to general topology, theory of sets.

University of Pittsburgh, June 9 to July 18, July 21 to August 29: Professor Knipp, differential equations; Professors Blumberg and Christiano, advanced calculus; Professor Bryson, vector analysis; Professor Taylor, functions of a complex variable; Professor

Laush, functions of a real variable; Professor Levine, modern algebraic theories; Professor Elyash, mathematical theory of probability. June 23 to August 15: Professor Elyash, differential equations; Professor Kovacs, mathematical theory of statistics; Professor Myers, recreational mathematics for teachers, theory of equations, teaching of secondary mathematics; Professor Leger, matrix theory. June 23 to August 15 (evenings): Professors Leger and Levine, differential equations; Professor Laird, mathematics of modern engineering; Professor Bryson, Laplace transform theory and applications.

University of Texas, June 10 to August 30: Professor Guy, Fourier and Laplace transforms. June 10 to July 21: Professor Craig, applications of tensor analysis; Professor Moore, theory of sets; Professor Ettlinger, research in differential and integral equations. July 22 to August 30: Professor Wall, theory of functions of real variables, functions of a complex variable; Professor Lane, continued fractions and applications.

University of Washington, June 23 to August 22: Professors Vaught, Pierce, and Blumenthal, linear algebra; June 23 to July 23: Professor Avann, introduction to modern algebra; Professor Livingston, advanced calculus I; Professors Blumenthal and Butler, differential equations; Professor Forrester, topics in applied analysis; Professor McFarlan, advanced analytic geometry; Professor Michael, topology. July 24 to August 22: Professor Beaumont, introduction to modern algebra; Professor Livingston, advanced calculus II; Professors Pierce and Haller, differential equations; Professor Forrester, topics in applied analysis; Professor Brownell, differential geometry; Professor Michael, topology.

University of Wisconsin, June 30 to August 22: Mr. Albright, theory and operation of computing machines; Visiting Lecturer Artzy, applied differential equations, theory of numbers; Professor Bicknell, introduction to mathematical statistics; Visiting Lecturer Dyer, advanced calculus (second half), advanced topics in algebraic topology; Dean Ingraham, determinants and matrices; Visiting Lecturer Dorothy Stone, advanced calculus (first half), theory of integration; staff, modern views of mathematics.

University of Wyoming, June 16 to July 18: Professor Barr, college geometry; Professor S. R. Smith, vector analysis, ordinary differential equations; Professor Steen, theory of equations. July 21 to August 22: Professor Barr, advanced calculus; Professor Schwid, solid analytic geometry, partial differential equations, Fourier series and boundary value problems; Professor S. R. Smith, numerical analysis.

West Virginia University, June 4 to July 16: Professor Cochran, astronomy for teachers; Professor Cunningham, higher plane curves; Mrs. Easton, history of mathematics; Professor Posey, advanced calculus; Professor Vest, complex variables. July 17 to August 27: Professor Cochran, advanced calculus; Mr. Lowenberg, geometry for teachers; Professor Peters, algebraic theories and higher plane curves; Professor Vest, complex variables. An Institute for High School Teachers of Mathematics and Physical Science will be conducted June 4 to July 16.

AEC-ASEE

Nine Institutes on Nuclear Energy for engineering educators will be held this summer under the sponsorship of the Atomic Energy Commission and the American Society for Engineering Education. The purpose of the institutes is to provide special training in the fields of nuclear energy and the nature of nuclear reactor problems so the teachers can incorporate this material in their teaching programs. The 1958 institutes will include four basic courses for teachers with no special background in nuclear energy, four advanced-level courses and one new basic course for teachers in technical institutes.

Participants will receive a minimum of two months' pay in addition to their regular salary for the academic year. For an applicant to be considered, the educational institution will be required to contribute to him a minimum of one month's salary. The AEC grant will match this contribution to a maximum of $750 plus round-trip railroad fare.

Applications for appointment may be obtained from the deans of engineering or from ASEE headquarters. They should be sent to Prof. W. Leighton Collins, Secretary of the ASEE, University of Illinois, Urbana, Illinois.

Each of the basic institutes will be combined programs of a university and a national laboratory, with a quota of from 25 to 30 participants at each location. The dates for all are June 23–August 15 and the locations are: North Carolina State College, Raleigh, with Oak Ridge National Laboratory; Cornell University, Ithaca, with Brookhaven National Laboratory; Purdue University, Lafayette, Ind. with Argonne National Laboratory; and University of California at Berkeley with Radiation Laboratory at Livermore.

The advanced institutes with quotas of from 20 to 25 participants will be as follows: Reactor Theory, University of Michigan, June 23 to August 15; Reactor Instrumentation and Control, Argonne National Laboratory, June 23 to August 15; Chemical Processing, Hanford Laboratories, June 22 to August 15; and Nuclear Metallurgy, Ames Laboratory, June 30 to August 22.

The course for teachers in technical institutes, with a quota of 25 participants, will be held at Pennsylvania State University from June 30 to August 8 and at Argonne National Laboratory from August 11 to August 22.

PERSONAL ITEMS

The establishment of the Richard Courant Lectureship in Mathematical Sciences at New York University was announced on January 8, 1958 at a convocation honoring Dr. Courant, Scientific Director of the University's Institute of Mathematical Sciences.

Hofstra College: Professor Loyal Ollman has been appointed Chairman of the Division of Natural Sciences, Mathematics, and Engineering; Dr. Harold Jacobson, Mr. Thadeus Smith, Mr. Abraham Soble and Miss Lois Chamberlin have been added to the Mathematics staff; Dr. E. R. Stabler will be on leave during the second semester of 1958 to promote a Peace by Education project.

Dr. W. B. Brown, formerly with Northrop Aircraft, Hawthorne, California, has been appointed Professor at Eastern Baptist College, St. Davids, Pennsylvania.

Mr. Horace Homesley, Jr., Graduate Student, University of Florida, has accepted a position on the staff of Atomic Power Division, Newport News Shipbuilding and Dry Dock Company, Virginia.

Dr. M. A. Hyman of Remington-Rand, UNIVAC, Philadelphia, has accepted a position as Senior Mathematician in the Research Center of the International Business Machines Corporation, Ossining, New York.

Dr. Azriel Rosenfeld, Senior Design Engineer, Ford Instrument Company, Long Island City, New York, has been appointed Visiting Assistant Professor at the Institute of Mathematics, Yeshiva University, New York City.

Professor Christopher R. Mitchell, Rhode Island College of Education, died on May 19, 1957.

Professor Emeritus W. A. Hurwitz, Cornell University, died on January 6, 1958. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE FORTY-FIRST ANNUAL MEETING OF THE ASSOCIATION

The forty-first annual meeting of the Mathematical Association of America was held at the University of Cincinnati and the Hotel Sheraton-Gibson, Cincinnati, Ohio on Thursday and Friday, January 30 and 31, 1958, in conjunction with the annual meeting of the American Mathematical Society. There were registered 750 persons, including 519 members of the Association.

Sessions of the Association were held on Thursday morning in the Ball Room of the Hotel Sheraton-Gibson and on Friday morning and afternoon in the Roof Foyer of the Hotel. President Price presided on Thursday morning and Friday afternoon, and Dean Mina S. Rees on Friday morning. The Program Committee for the meeting consisted of E. A. Cameron, Chairman; E. P. Northrop, and D. E. Richmond.

FIRST SESSION OF THE ASSOCIATION

Retiring Presidential Address: "Mathematics Courses for Mathematics Teachers," by Dean W. L. Duren, Jr., University of Virginia.

"Lectureship Programs of the Association." Panel: Professor B. W. Jones, University of Colorado, Chairman; Professor G. B. Price, University of Kansas; Professor R. D. Schafer, University of Connecticut; Professor J. C. Eaves, University of Kentucky.

SECOND SESSION OF THE ASSOCIATION

"The National Science Foundation and Institutes for Mathematics Teachers," by Dr. Harry Kelly of the National Science Foundation.

"Organization and Operation of Institutes." Panel: Professor L. W. Johnson, Oklahoma State University, Chairman; Professor H. M. Bacon, Stanford University; Professor E. A. Cameron, University of North Carolina; Professor H. F. Fehr, Columbia University; Professor Joseph Landin, University of Illinois; Professor D. E. Richmond, Williams College; Professor R. M. Thrall, University of Michigan; Professor Henry Van Engen, University of Wisconsin.

THIRD SESSION OF THE ASSOCIATION

Annual Business Meeting of the Association.

"Relation of Institutes to the work of the Commission on Mathematics," by Dean A. E. Meder, Jr., Rutgers University, Executive Director of the Commission on Mathematics.

"Appropriate Courses for Institutes." Panel: Professor R. E. Johnson, Smith College; Professor R. A. Rosenbaum, Wesleyan University; Professor George Pólya, Stanford University (paper read by Professor Bacon); Professor E. J. McShane, University of Virginia; Professor J. L. Snell, Dartmouth College.

"Follow-Up Programs to Institutes," by Mrs. Marie S. Wilcox, Thomas Carr Howe High School, Indianapolis, Indiana.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Thursday afternoon in Parlor H of the Hotel Sheraton-Gibson with twenty-three members present. Among the more important items of business transacted were the following:

The Board approved the appointment by President Price of the following Nominating Committee for 1958: Wallace Givens, Chairman, H. M. Bacon and Rothwell Stephens.

The Board elected Lloyd J. Montzingo, Jr., of the University of Buffalo as Associate Secretary for 1958–1962, and E. A. Cameron of the University of North Carolina as member of the Finance Committee for 1958–1961.

The Board approved the following schedule of future meetings: Massachusetts Institute of Technology, August 25–26, 1958; University of Pennsylvania, January 22–23, 1959; University of Utah, August 31–September 1, 1959; and Oklahoma State University, August 28–29, 1961 (sic).

The Board ratified the constitution and by-laws of the Conference Organization of the Mathematical Sciences, which is replacing the Policy Committee for Mathematics.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION

The annual business meeting of the Association was held on Friday, January 31, 1958 in the Roof Foyer of the Hotel Sheraton-Gibson, Cincinnati, Ohio. President G. B. Price presided.

The Secretary announced the results of the balloting for officers in which 2370 votes were cast: Professor G. B. Thomas, Jr., Massachusetts Institute of Technology was elected First Vice-President for the two-year term 1958–1959, and Professors Howard Eves, University of Maine, and J. S. Frame, Michigan State University, were elected Governors for the three-year term 1958–1960.

The membership of the Association was 7094 at the end of 1957, a gain of 603 during the year.

Reports were made by Professor B. W. Jones for the Committee on Visiting Lecturers and by Professor J. S. Frame for the Committee on Employment Opportunities.

MEETING OF OTHER ORGANIZATIONS

The American Mathematical Society held its sessions from Tuesday, January 28 through Thursday. The Gibbs lecturer was Professor H. J. Muller. Invited addresses were delivered by Professor Nelson Dunford and Dr. C. D. Papakyriakopoulos. The retiring presidential address of Professor R. L. Wilder was delivered on Wednesday afternoon on the campus of the University of Cincinnati.

ARRANGEMENTS, ENTERTAINMENT, AND RECREATION

The Committee on Arrangements for the meeting consisted of: I. A. Barnett, Chairman; R. W. Allen, J. L. Baker, Jr., H. M. Gehman, Arno Jaeger, H. D. Lipsich, C. I. Lubin, G. M. Merriman, B. A. Raymond, W. E. Restemeyer, J. W. T. Youngs.

Registration headquarters was on the mezzanine of the Hotel. Sleeping accommodations were available in the Hotel Sheraton-Gibson and other nearby hotels. Meals were available in the Hotel and at numerous nearby restaurants and cafeterias. The employment register and a text book exhibit were located near the registration desk.

A tea for the attending mathematicians and their guests was given by the University of Cincinnati on Wednesday afternoon in the Union Building on the Campus. The University provided special bus service on Wednesday from the Hotel to the Campus and return.

A complimentary banquet, through the generosity of the companies listed below, was held on Thursday evening in the Roof Garden of the Hotel. The sponsors of the banquet were AVCO Corporation, Cincinnati Gas and Electric Co., Cincinnati Milling Machine Co., General Electric Co. (Aircraft Gas Turbine Division), International Business Machines Corporation, Procter and Gamble Co., Union Central Life Insurance Co., Western and Southern Life Insurance Co. Professor T. H. Hildebrand served as toastmaster. Speakers were Dean H. S. Greene and Professor Emeritus C. N. Moore of

the University of Cincinnati, President Richard Brauer of the Society, and President G. B. Price of the Association. Professor G. A. Hedlund presented a resolution expressing the warm appreciation of those present to those who by their efforts and contributions have aided in making the meetings effective and enjoyable.

HARRY M. GEHMAN, *Secretary-Treasurer*

THE OCTOBER MEETING OF THE MINNESOTA SECTION

The annual fall meeting of the Minnesota Section of the Mathematical Association of America was held on October 5, 1957 at Mankato State College, Mankato, Minn. Professor W. J. Thomsen of Mankato presided at the morning session. The Section Secretary, Professor F. L. Wolf of Carleton College, presided over the afternoon session in place of the Section Chairman, Professor O. E. Stanaitis of St. Olaf College, who was ill. There were 44 persons registered, including 28 members of the Association.

At the business meeting, Professor G. K. Kalisch of the University of Minnesota, Chairman of the Sectional Committee for the High School Contest, reported on the activities of his committee. Professor F. L. Wolf reported on the meeting of section officers held at the summer meeting of the Association in Pennsylvania. It was moved, seconded, and passed that the Section chairman be directed to appoint a Committee on High School-College Relations to investigate and implement ways and means by which the Section may have influence in bettering the mathematical education of students at the high school level in the Minnesota region.

The invitational address was given by Professor L. Hurwicz of the University of Minnesota. The address was entitled *Some Recent Developments in Mathematical Economics and Econometrics.*

The following short papers were presented:

1. *Some remarks on the new high school mathematics curriculum developed at the University of Illinois*, by Professor R. W. Sloan, Carleton College.

The author discussed some of the points wherein the U.I.C.S.M. Program differs from traditional courses at the high school level.

2. *The Apollonius Circle-Contact Problem*, by Dr. C. N. Mills, Sioux Falls College.

Historically the Apollonius Circle-Contact Problem is concerned with the construction of circles tangent to three circles mutually tangent externally. This paper gave the ten different cases from three points, or null circles, to three circles mutually external, but not tangent. The different cases have been constructed making use of many theorems of modern geometry. For this last case, the radius of each of the eight possible circles has been determined recently by Beckman Martin of Toledo, Ohio, using a trigonometric method. As far as the writer knows, no eighth degree algebraic equation has been published which gives the radius of each of the eight circles.

3. *Automatic coding for digital computers*, by Zane Motteler, University of Minnesota.

A good example of automatic coding in use today in large digital computers is the FORTRAN system for the IBM 704. In this system, mathematical formulas punched on cards in a definite format are translated directly into binary code by the machine, with the aid of a long binary tape. Basic problems of machine logic such as setting aside of storage space, loop calculations, subroutines and conditional transfers are very ingeniously handled. Automatic coding is especially advantageous in terms of the time saved in learning the code and in the initial coding of a problem. Input and output is extremely flexible. However, debugging time and printing time are often quite long, and make up for the other savings. Storage is inflexible and often uneconomical; spaces are wasted unless the needs of the problem are constant and can be set exactly in advance. Generally, however, automatic coding is preferred by coders and machine operators to standard symbolic coding because of its simplicity and the excellent output available.

F. L. WOLF, *Secretary*

THE OCTOBER MEETING OF THE OKLAHOMA SECTION

The fall meeting of the Oklahoma Section of the Mathematical Association of America was held at Oklahoma City University, Oklahoma, on October 25, 1957. Professor W. A. Rutledge, Vice-Chairman of the Section, presided. There were 192 persons in attendance, including 63 members of the Association.

The following officers were elected for one-year terms: Chairman, Professor W. A. Rutledge, The University of Tulsa; Vice-Chairman, Professor Eunice Lewis, The University of Oklahoma; Secretary-Treasurer, Professor R. V. Andree, The University of Oklahoma. Professor R. B. Deal, Oklahoma State University of Agriculture and Applied Science, was appointed representative from the section on The Oklahoma Junior Academy of Science Advisory Committee. The committee for the 1958 High School Mathematics Contest was appointed as follows: Professor H. N. Carter, The University of Tulsa, Chairman; Miss Sarah Burkhart, The University of Tulsa; Mr. Richard G. Laatsch, The University of Tulsa.

After a discussion of the recommendations of the Commission on Mathematics of the C.E.E.B., and of the various topics considered at the August meeting of Sectional officers of M.A.A., it was decided to investigate the possibility of providing a limited "visiting lecture service" for high schools in this area. A committee consisting of: Professor Kathrine Mires, Northwestern State College (Chairwoman); Mr. Thomas Hill, Classen High School, Oklahoma City; and Professor R. B. Deal, Oklahoma State University of Agriculture and Applied Science was appointed.

The fall meeting of the Oklahoma Section is held in conjunction with the Oklahoma Education Association and is devoted to papers of particular interest to high school teachers. Research papers are presented in the spring meeting.

The morning meeting consisted of two symposia devoted to *Coming Practices in High School and College Mathematics*, and an hour talk by Professor Gertrude Hendrix, University of Illinois Committee on School Mathematics, on the topic, *The Mathematics Project of the University of Illinois Committee on School Mathematics: Content, Method, and Modus Operandi.*

Discussion in the symposia was spirited and varied with considerable audience participation. The invited speakers were: Eunice Lewis, University High School, Norman; Roy B. Deal, Oklahoma State University of Agriculture and Applied Science, Stillwater; Thomas Hill, Classen High School, Oklahoma City; William N. Huff, The University of Oklahoma, Norman; E. Truman Wester, Central State College, Edmond; and W. R. Orton, The University of Arkansas, Fayetteville; Richard Johnson, Smith College, Northampton, Mass.; Kathrine C. Mires, Northwestern State College, Alva; W. A. Rutledge, The University of Tulsa, Tulsa.

Some of the points made by various speakers were:

Several high schools in the state are currently giving good courses in freshman college mathematics and the universities are responding by permitting well-prepared incoming freshmen to skip the first semester or even the first year of mathematics. After 1959–60, remedial mathematics (Elementary Algebra and Geometry), will not be given at The University of Oklahoma, excepting in summer school. A sharp increase in enrollments in college mathematics was noted in most institutions. One school reported almost 300% increase this term. It was pointed out that, as far as general academic preparation is concerned, the state of Oklahoma is, at least on paper, better situated than most states since 95% of all teachers (grade school, junior high school and high school) have received bachelor's degrees from recognized institutions, and 40% have master's degrees. Relatively few of these degrees are in mathematics (as opposed to mathematics education). However, the belief was expressed that five years of training (rather than the present four) may soon be required for the standard (five year) teaching certificate in mathematics and science. Oklahoma no longer issues "life certificates." Discussion was lively

concerning the need for competent mathematicians to help develop courses for high school teachers which would cut across the boundaries of the usual graduate courses and present in an integrated course, the aspects of modern mathematics most needed by high school teachers. Several speakers urged the requirement of more modern abstract algebra.

The general consensus seemed to be that the mathematical outlook in the state of Oklahoma was encouraging, and that there was promise of further improvement.

R. V. ANDREE, *Secretary*

THE JANUARY MEETING OF THE NORTHERN CALIFORNIA SECTION

The twentieth annual meeting of the Northern California Section of the Mathematical Association of America was held at San Francisco State College, January 18, 1958. Professor B. J. Lockhart, Acting Chairman of the Section, presided at both the morning and afternoon sessions. There were 146 persons in attendance, including 80 members of the Association.

At the business meeting the following officers were elected for the coming year: Chairman, Professor B. J. Lockhart, U. S. Naval Postgraduate School; Vice-Chairman, Professor Gerald Preston, San Jose State College; Secretary-Treasurer, Professor Roy Dubisch, Fresno State College. The section also voted to continue the high school lectureship program in 1958.

By invitation of the section, Professor George Forsythe, Stanford University, delivered an address at the morning session entitled *Computers and Numerical Analysis in Mathematics Education*. Abstract of this address follows:

A prime need of students of mathematics is to learn that mathematics is a live and interesting field, to which the student himself can make an active contribution. Digital computers, from the hand-cranked model to the most advanced electronic machine, can provide better students with that feeling of life and interest which so fascinates them. The hand-cranked model vastly increases the scope of the arithmetic which a junior high-school student can perform, while the high school student can easily learn to code a machine like the IBM 650. In college a generous admixture of the numerical point of view livens up any course in mathematics. The necessity of actually getting a numerical answer out of a machine will increase anyone's respect for careful analysis. Numerical analysis is defined and discussed. As an example suited to high school or college students, the Seidel iterative solution of a system of linear algebraic equations is illustrated, together with a procedure for accelerating the convergence.

The following papers were presented:

1. *An objectionable quadrangle theorem*, by Professor C. M. Fulton, University of California, Davis.

The following theorem is frequently found in analytic geometry texts: The midpoints of two opposite sides of any quadrilateral and the midpoints of the diagonals are the vertices of a parallelogram. This theorem is not true for a trapezoid or parallelogram. Some suggestions are made for the improvement and proper interpretation of theorems of similar type.

2. *On the Brocard points of a triangle*, by Professor P. Yff, Fresno State College.

Given any triangle and any point in the Euclidean plane and a fixed angle θ, they determine a path which converges to the perimeter of a unique triangle and independent of the given point. Convergence occurs only for a certain set of values of θ. Each side of the second triangle forms the angle θ with the corresponding side of the given triangle. Each Brocard point of the given triangle is the limit point of a sequence of such inscribed triangles, the result being independent of θ.

3. *Reports on contests and lectureships*, by Professor E. M. Beesley, University of Nevada; Professor D. W. Blakeslee, San Francisco State College; and Professor Roy Dubisch, Fresno State College.

Professor Beesley reported that the Mathematics Department of the University of Nevada, with the assistance of the Nevada Chapter of Pi Mu Epsilon, held its first annual high school prize examination in mathematics on April 7, 1957. The examination was prepared and administered by the university's mathematics staff. Two hundred ninety-one students took the examination. Cash prizes, medals, and certificates were awarded.

Professor Blakeslee reported that 112 high schools and 3300 students participated in the 1957 Mathematical Association of American contest as compared to 73 schools and 2300 students in 1956.

Professor Dubisch reported on the first visiting lectureship program for high schools sponsored by the section. Not as many high schools participated as had been anticipated but the reaction of those who did was most favorable.

4. *Some properties of a generalized Euler ϕ-function*, by Professor H. L. Alder, University of California, Davis.

It is shown that many of the well-known properties of the Euler ϕ-function also hold with some modification for the generalized Euler ϕ-function $\phi(n, m)$, which was recently introduced by the author (see "A generalization of the Euler ϕ-function," to appear soon in this MONTHLY).

5. *On the digital control of machine tools*, by Professor H. D. Huskey, University of California, Berkeley.

The high performance of present day aircraft requires the precise machining of airframe structures merely to control weight. The specification of the motion of the parts of a milling machine to produce a given cutting path is an interesting problem in three dimensional analytic geometry. Methods used to cut required contours and surfaces are discussed in detail and ramifications in design techniques and engineering drawing are mentioned.

6. *On the curriculum for prospective high school teachers*, by Professor G. Pólya, Stanford University.

The author reports on the aims and contents of lectures he gave, or is giving, in three different institutes for high school teachers. This report will appear in the MONTHLY.

7. *Geometry as a way of thinking*, by Dr. E. Greer, Lockheed Aircraft Corporation, Sunnyvale.

The declining enrollment in geometry in the secondary schools is, in part, due to the way instructors are presenting the subject matter. The prime importance of this course is to train the student to make conclusions through scientific reasoning, developed so easily through the demonstrative proofs used in its exercises. The facts learned are secondary in importance. To increase, only slightly, the percentage of correct decisions made by people by this method of thinking would be valuable to society. The purpose of this paper is to point out this important function of geometry and to encourage improvement in its presentation.

8. *On looking backward at solutions—some remarks and examples*, by Professor C. M. Larsen, San Jose State College.

A student of mathematics can learn a great deal by examining his work after reaching a solution to a problem, instead of abandoning the problem as "done." Examining the solution, however, means more than merely gazing at it. Specifically, checking the solution and variation of the problem are among the useful activities that should enter into "Looking Back." These activities were illustrated in connection with two examples from elementary calculus, and the belief was expressed that such efforts can add considerable interest to otherwise routine problems.

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CALENDAR OF FUTURE MEETINGS

Thirty-ninth Summer Meeting, Massachusetts Institute of Technology, Cambridge, Massachusetts, August 25–28, 1958.

Forty-second Annual Meeting, University of Pennsylvania, Philadelphia, Pennsylvania, January 22–23, 1959.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Washington and Jefferson College, Washington, Pennsylvania, May 3, 1958.

ILLINOIS, Illinois College, Jacksonville, May 9–10, 1958.

INDIANA, Ball State Teachers College, Muncie, May 3, 1958.

IOWA, Drake University, Des Moines, April 18, 1958.

KANSAS, Kansas State Teachers College, Emporia, April 12, 1958.

KENTUCKY, University of Kentucky, Lexington, April 26, 1958.

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MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Randolph-Macon Woman's College, Lynchburg, Virginia, April 26, 1958.

METROPOLITAN NEW YORK, Hofstra College, Hempstead, New York, April 19, 1958.

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MINNESOTA, St. John's University, Collegeville, Minnesota, May 17, 1958.

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NEBRASKA, University of Nebraska, Lincoln, April 19, 1958.

NEW JERSEY, Rutgers University, New Brunswick, November 1, 1958.

NORTHEASTERN

NORTHERN CALIFORNIA, University of California, Berkeley, June 17, 1958 (joint meeting with ASEE, mathematics division).

OHIO, Denison University, Granville, April 26, 1958.

OKLAHOMA, Central State College, Edmond, April 18–19, 1958.

PACIFIC NORTHWEST, Oregon State College, Corvallis, June 20, 1958.

PHILADELPHIA, Lehigh University, Bethlehem, November 29, 1958.

ROCKY MOUNTAIN, Colorado State College, Greeley, May 9–10, 1958.

SOUTHEASTERN

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SOUTHWESTERN, University of New Mexico, Albuquerque, April 11–12, 1958.

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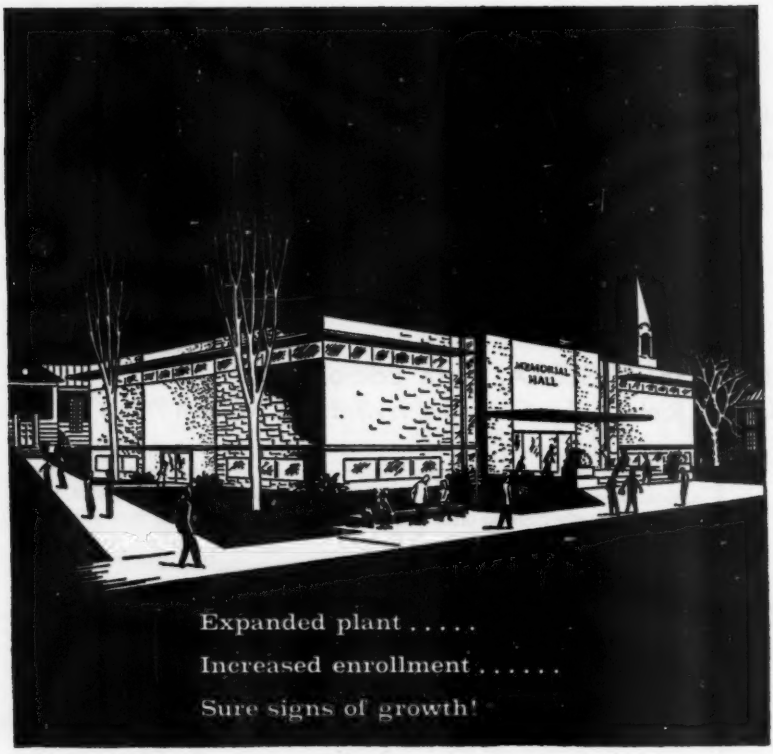

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